The Risk of Out-of-Sample Portfolio Performance*

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May 14, 2022

Abstract

We show theoretically and empirically that estimated portfolios bear substantial out-of-sample utility risk in high-dimensional settings or when these portfolios exploit in-sample near-arbitrage opportunities. We use our novel analytical characterization of out-of-sample utility risk to propose a robustness measure that balances out-of-sample utility mean and volatility. We demonstrate that while individual portfolios do not offer maximal robust performance, portfolio combinations achieve the optimal tradeoff between out-of-sample utility mean and volatility and are more resilient to estimation errors. Our analysis of out-of-sample performance risk has implications for constructing and evaluating quantitative investment strategies and models of the stochastic discount factor.

Keywords: parameter uncertainty, mean-variance portfolio, shrinkage.
JEL Classification: G11, G12

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1 Introduction

How can one construct investment strategies that deliver an optimal risk-return tradeoff? Markowitz (1952) addresses this fundamental question in finance by proposing a simple mean-variance mathematical program to construct well-diversified portfolios. A critical assumption in Markowitz's theory is that investors know the true distributional properties of stock returns. This assumption is problematic because, since Markowitz (1952), academics have extensively documented that sample mean-variance portfolios contaminated by the estimation errors in the vector of means and the covariance matrix of stock returns tend to deliver poor out-of-sample performance (Jagannathan and Ma, 2003; DeMiguel, Garlappi, and Uppal, 2009b; Tu and Zhou, 2011). A key insight from this literature is that out-of-sample portfolio performance is a random variable influenced by the random sample used to construct the portfolio. An influential paper that explicitly acknowledges the stochastic nature of out-of-sample portfolio performance is Kan and Zhou (2007). Kan and Zhou characterize the *average* out-of-sample utility (OOSU) of mean-variance investors and exploit this analytical characterization to assess the average utility losses of sample portfolios and construct portfolio combinations that improve OOSU mean.

Instead, in this manuscript, we study the OOSU *risk* of sample portfolios, which is essential to fully understand the stochastic nature of the performance of quantitative investment strategies. In particular, we characterize the asymptotic distribution of the OOSU of the sample mean-variance (SMV) portfolio, the sample global-minimum-variance (SGMV) portfolio, and any combination of the two portfolios. We show that the asymptotic distribution of OOSU is Gaussian, and thus it is fully characterized by its mean and *variance*, which we characterize in closed form. We then derive the analytical expression for the finite-sample OOSU variance of the SMV and SGMV portfolios and any combination of the two.

In line with the idea that the quality of investment strategies should not be judged by a single realization of their *stochastic performance* (Lo, 2002; Harvey and Liu, 2014), our closed-form expressions of the OOSU variance of sample portfolios advances our understanding of estimation risk in portfolio selection by providing a more comprehensive picture of the performance uncertainties faced by investors exploiting quantitative strategies. For example, the OOSU two-sigma interval of the SMV portfolio calibrated from a dataset of 25 portfolios of stocks sorted on size and book-to-market (25SBTM) with 120 monthly return observations is $[-12.7\%, 0.66\%]$ for a risk-aversion coefficient of three. This large variation in the monthly OOSU of quantitative strategies represents a paramount concern for investors who deem performance uncertainty a critical factor affecting their investment decisions.

We use our characterization of OOSU risk to develop a *novel* metric of portfolio robustness defined as the difference between OOSU mean and a multiple of OOSU risk. We assess the robustness of quantitative strategies that combine the SMV and SGMV portfolios as in Garlappi, Uppal, and Wang (2007) and Kan, Wang, and Zhou (2021b).[1] We show theoretically that neither the SMV portfolio nor the SGMV portfolio offers the maximal robust performance individually and that one must *optimally* combine both to achieve a better tradeoff between OOSU mean and OOSU volatility.

The robustness criterion we propose to combine portfolios resembles the diversification idea behind the mean-variance efficient frontier of Markowitz (1952). Instead of obtaining the combination of stocks that achieves the optimal tradeoff between mean return and variance, we obtain the combination of estimated portfolios that achieves the optimal tradeoff between OOSU mean and variance. Our empirical analysis shows that investment strategies that optimize our robustness metric deliver better out-of-sample performance than those that ignore OOSU risk across different datasets.

Our manuscript makes four contributions to the existing literature on parameter uncertainty and portfolio selection. First, we characterize the asymptotic distribution and the finite-sample variance of the OOSU of the SMV portfolio, the SGMV portfolio, and any combination of these two portfolios. Using our analytical results, we document that the SMV portfolio's OOSU volatility is substantially larger than that of the SGMV portfolio. Take, for instance, the 25SBTM dataset and a risk-aversion coefficient of three. For this case, the OOSU standard deviation of the SMV portfolio is 29 times larger than that of the SGMV portfolio when both portfolios are estimated using 120 monthly observations. In this particular case, the SMV portfolio requires an unrealistically large sample size of over 13,000

[1] Our portfolio robustness metric can accommodate a broader range of portfolio combinations. Indeed, in Section IA.4.5 of the Internet Appendix, we extend our analysis to a shrinkage portfolio that combines the SMV portfolio, the SGMV portfolio, and the equally weighted portfolio as in Tu and Zhou (2011).

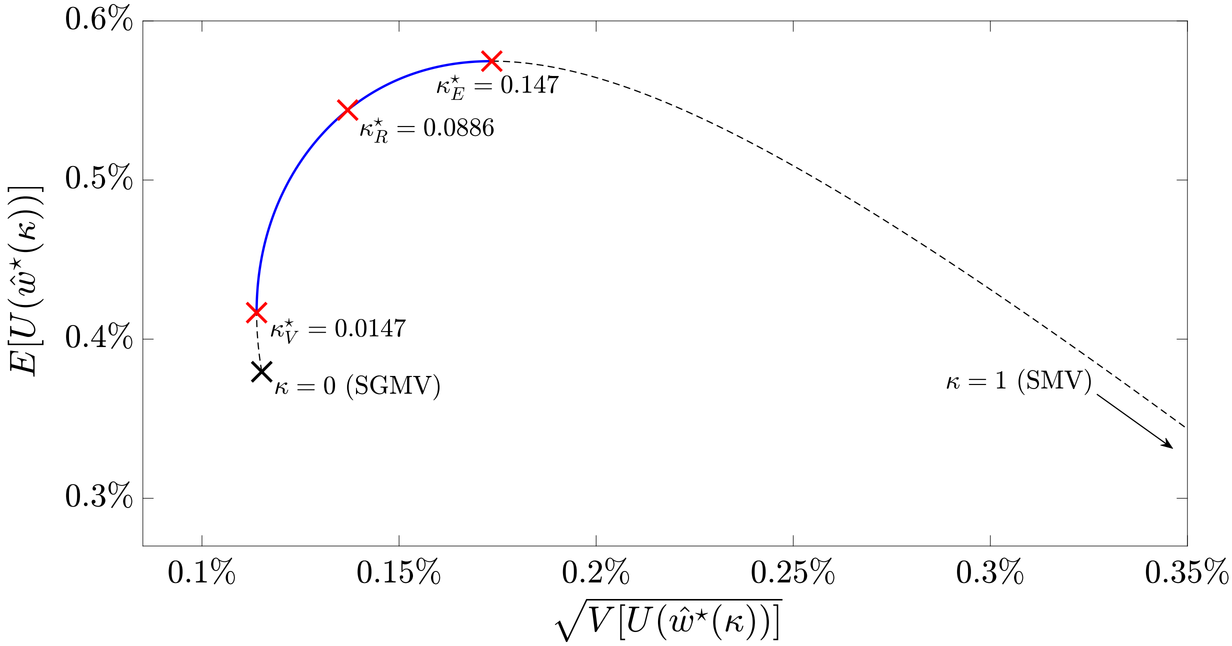
monthly observations –more than 1,000 years of data– to deliver an out-of-sample performance as stable as that of the SGMV portfolio. More generally, we show that the OOSU risk of sample portfolios increases in high-dimensional settings where the number of assets is large relative to the number of sample observations.

Our second contribution is to propose a novel measure of portfolio robustness defined as the difference between OOSU mean and a multiple of OOSU standard deviation. In our view, a robust portfolio should deliver a stable out-of-sample utility that performs well on average. The robustness measure we propose is in the spirit of this view, and we show that it asymptotically corresponds to the Value-at-Risk of out-of-sample utility. We then analyze how portfolio robustness responds to the presence of near-arbitrage opportunities, which Kozak, Nagel, and Santosh (2018) define as low-variance principal components that contribute substantially to the maximum squared Sharpe ratio. We theoretically demonstrate that in-sample near-arbitrage opportunities and OOSU risk are positively correlated, indicating that portfolios that deliver high in-sample Sharpe ratios may not be robust.

In addition, the relation between OOSU risk and near-arbitrage opportunities has implications for the evaluation of quantitative strategies. Our theory shows that sample strategies exploiting low-variance principal components to achieve large Sharpe ratios are inherently riskier due to their larger OOSU risk. This insight is distinct from Harvey and Liu (2014) who point out that high-Sharpe-ratio trading strategies may exist by chance and not because they truly provide investors with a skill that will continue to outperform out of sample. In contrast, our theory does not rule out the existence of high-Sharpe-ratio quantitative strategies, but it highlights the important caveat that OOSU risk increases for strategies that exploit *in-sample* near-arbitrage opportunities.

Our third contribution is to propose a novel calibration criterion that exploits our portfolio robustness measure for combining the SMV portfolio with the SGMV portfolio. Using the analytical characterization of OOSU mean and volatility for shrinkage portfolios, we show that neither the SMV portfolio nor the SGMV portfolio delivers maximal robustness individually, and one needs to combine both portfolios to attain an optimal tradeoff between OOSU mean and volatility. We demonstrate that our robust portfolio assigns a larger tilt toward the SGMV portfolio than the shrinkage portfolio maximizing OOSU mean. This larger tilt

Figure 1: Out-of-sample utility efficient frontier



Notes. This figure depicts the out-of-sample utility mean and standard deviation of shrinkage portfolios $\hat{w}^\star(\kappa)$ that combine the sample global-minimum-variance (SGMV) and sample mean-variance (SMV) portfolios for different values of κ. The shrinkage intensity $\kappa = 0$ corresponds to the SGMV portfolio, and $\kappa = 1$ to the SMV portfolio. The population vector of means and covariance matrix of stock excess returns are calibrated from the monthly return data of the 25 portfolios of stocks sorted on size and book-to-market. The shrinkage portfolios are estimated using a sample size of $T = 120$ months and a risk-aversion coefficient of $\gamma = 3$. The solid blue line corresponds to the efficient tradeoff between out-of-sample utility mean and standard deviation provided by the shrinkage portfolios whose shrinkage intensity κ is in the interval $[\kappa_V^\star, \kappa_E^\star]$. The shrinkage intensity $\kappa_R^\star$ maximizes the portfolio robustness measure in Section 6 with $\lambda = 2$.

toward the SGMV portfolio allows our robust portfolio to achieve a substantially more stable out-of-sample performance while sacrificing only a small out-of-sample average performance.

Figure 1 illustrates the main idea of our proposed method. The vertical axis depicts the OOSU mean of the shrinkage portfolio, and the horizontal axis depicts the OOSU standard deviation. The parameter κ is the shrinkage intensity that determines the combination between the SMV and SGMV portfolios. The shrinkage portfolio exploiting $\kappa_E^\star$ maximizes OOSU mean and the shrinkage portfolio exploiting $\kappa_R^\star$ maximizes our proposed measure of portfolio robustness. Figure 1 shows that the proposed robust shrinkage portfolio exploiting $\kappa_R^\star$ decreases OOSU standard deviation by 21% at the expense of only a 5% reduction in OOSU mean relative to the shrinkage portfolio exploiting $\kappa_E^\star$.[2]

[2]While Figure 1 considers the true shrinkage intensities, our simulation results show that our proposed shrinkage intensity $\kappa_R^\star$ delivers a more significant improvement in portfolio performance compared to $\kappa_E^\star$ when the shrinkage intensities are unknown and must be estimated.

Our fourth contribution is to evaluate the out-of-sample performance of our proposed robust portfolio relative to several benchmarks. Our simulations show that the robust shrinkage portfolio delivers a better tradeoff between OOSU mean and standard deviation than the portfolio maximizing only OOSU mean. An appealing feature of our method is that the robust shrinkage portfolio also delivers a larger OOSU mean than the portfolios specifically designed to maximize OOSU mean in cases where returns are not Gaussian and shrinkage intensities are estimated from the data. This result indicates that our proposed portfolio framework is resilient to estimation errors and model misspecification.

In addition, our simulations confirm our theoretical result that the OOSU risk of sample portfolios increases when these portfolios exploit in-sample near-arbitrage opportunities. In particular, we show that sample portfolios that exploit all low-variance principal components command a substantially larger OOSU risk than portfolios using shrinkage covariance matrices, which attenuate the impact of low-variance principal components on the performance of estimated portfolios. Inspired by this insight, we employ shrinkage covariance matrices in the construction of the benchmark strategies and our proposed shrinkage portfolio in the performance analysis with empirical data.

We study the performance of our robust portfolio on six empirical datasets of monthly return data. We document that the proposed robust shrinkage portfolio outperforms in terms of certainty-equivalent return, which corresponds to the empirical out-of-sample utility of estimated portfolios, and Sharpe ratio. Specifically, for an estimation window of 120 monthly observations the median improvement in terms of certainty-equivalent return (Sharpe ratio) net of transaction costs across the six datasets is 79% (29%) relative to the shrinkage portfolio only maximizing OOSU mean, 151% (30%) relative to the SMV portfolio, 50% (21%) relative to the SGMV portfolio, 100% (51%) relative to the timing strategy of Kirby and Ostdiek (2012), and 179% (74%) relative to the equally weighted portfolio. The outperformance of the robust shrinkage portfolio relative to the benchmark portfolios is similar in magnitude in the absence of transaction costs. In addition, we use three-year non-overlapping windows to gauge the stochastic nature of the out-of-sample performance of the considered shrinkage portfolios. We find that the out-of-sample certainty-equivalent return of our robust portfolio is larger on average and more stable over time than that of the shrinkage portfolio only maximizing OOSU mean.

Our theory also has implications for the evaluation of asset pricing models that define the stochastic discount factor (SDF) as a linear combination of test assets. In particular, Section IA.2 of the Internet Appendix shows that there is a link between the OOSU of shrinkage portfolios and the out-of-sample R-squared of a particular robust SDF model. Therefore, our analytical characterization of OOSU risk can be applied to assess the uncertainty of the out-of-sample R-squared of SDF models. Our theory suggests that SDF models constructed from many test assets with short time series and that capture near-arbitrage opportunities will deliver, in general, unreliable results due to their large out-of-sample R-squared risk. In contrast, SDF models built from our robust approach will deliver more reliable results with lower out-of-sample R-squared risk.

Our manuscript highlights the value of accounting for out-of-sample performance risk when evaluating and constructing quantitative investment strategies. On the evaluation front and in line with the idea that investment strategies should not be judged solely based on a single past performance realization, financial institutions can use our theory to provide investors valuable information about performance uncertainty. On the construction side, we show that sample portfolios that exploit near-arbitrage opportunities in high-dimensional settings are doomed to experience large OOSU risk. In contrast, portfolios that account for OOSU risk are more resilient to estimation errors and deliver a more robust performance.

2 Literature review

We build on the literature pioneered by Kan and Zhou (2007) who study the *average* out-of-sample utility losses of mean-variance portfolios. Kan and Zhou use their analytical characterization of OOSU mean to build shrinkage portfolios that mitigate the impact of parameter uncertainty on performance.[3] Inspired by the work of Kan and Zhou, an extensive literature studies the OOSU mean of different sample portfolios to propose new strategies that

[3]Earlier studies consider out-of-sample utility mean as a portfolio-choice criterion under parameter uncertainty using a Bayesian framework, such as Brown (1976) and Frost and Savarino (1986). However, Kan and Zhou (2007) are the first to analytically characterize the average out-of-sample utility losses from parameter uncertainty under the assumption of Gaussian returns.

improve performance in the presence of parameter uncertainty.[4] In contrast to these papers, our work theoretically characterizes the OOSU *standard deviation* of sample portfolios and highlights the relevance of accounting for out-of-sample performance risk in evaluating and constructing quantitative investment strategies and asset pricing models.

Our work is also related to several papers that study the *distribution* of out-of-sample portfolio performance measures. For instance, Kan and Smith (2008) and Kan et al. (2021b) derive the distribution of the out-of-sample mean return and variance of efficient portfolios, Kan, Wang, and Zheng (2021a) derive the distribution of the out-of-sample Sharpe ratio of the sample tangency portfolio, and Yuan and Zhou (2022) derive the distribution of the out-of-sample Sharpe ratio of several portfolio combinations. We complement these papers by deriving the asymptotic OOSU distribution of any combination between the SMV and SGMV portfolios. We also characterize the finite-sample OOSU variance of any combination of the SMV, SGMV, and equally weighted portfolios. Moreover, unlike these papers, we use the OOSU mean and volatility to measure the robustness of sample portfolios and derive an optimal robust shrinkage portfolio. To the best of our knowledge, our work is the first to exploit out-of-sample performance volatility to construct quantitative investment strategies.

In the portfolio construction front, our work is closely related to the literature that exploits *shrinkage estimators* to mitigate the impact of parameter uncertainty.[5] Such estimators are traditionally applied to alleviate the impact of parameter uncertainty affecting the inputs of the portfolio problem, like the mean (Jorion, 1986; Barroso and Saxena, 2021) and the covariance matrix (Ledoit and Wolf, 2003, 2004, 2017, 2020). Unlike these papers, we focus on combining *portfolios* to attain an optimal tradeoff between OOSU mean and volatility.

The shrinkage portfolios we consider in this manuscript share fundamental elements with regularization, which is one of the most common machine learning approaches adopted in

[4]See Zhou (2008), DeMiguel et al. (2009b), Frahm and Memmel (2010), Tu and Zhou (2011), DeMiguel, Martín-Utrera, and Nogales (2013a, 2015), Branger, Lučivjanská, and Weissensteiner (2019), Kircher and Rosch (2021), Füss, Koeppel, and Miebs (2021), Kan and Wang (2021), and Kan et al. (2021b).

[5]A large number of papers propose different approaches to alleviate parameter uncertainty in portfolio selection using, e.g., Bayesian statistics (Jorion, 1986; Avramov and Zhou, 2010), factor models (De Nard, Ledoit, and Wolf, 2019), forward-looking information (DeMiguel, Plyakha, Uppal, and Vilkov, 2013b), model misspecification (Rapponi, Uppal, and Zaffaroni, 2021), robust optimization (Goldfarb and Iyengar, 2003), sparse estimation (Goto and Xu, 2015; Ao, Li, and Zheng, 2019), and weight constraints (Jagannathan and Ma, 2003; DeMiguel, Garlappi, Nogales, and Uppal, 2009a; Olivares-Nadal and DeMiguel, 2018).

the recent asset pricing literature (Giglio, Kelly, and Xiu, 2021; Bryzgalova, Pelger, and Zhu, 2021). Like regularization, the shrinkage portfolio considered in this manuscript helps mitigate the impact of sampling volatility on the estimated portfolio weights. We show that our robust approach to shrinkage portfolios refines and improves the out-of-sample performance of investment strategies over existing methods.

The shrinkage portfolio approach is not only a practical technique for alleviating the impact of statistical errors on the performance of estimated portfolios, but it is also an economically sound method related to the investment-decision problem of ambiguity-averse investors. In particular, in Section IA.1 of the Internet Appendix, we characterize the exact relationship between the shrinkage portfolio that combines the SMV and SGMV portfolios and the ambiguity-averse portfolios considered by Garlappi et al. (2007). We show in closed form that a larger degree of ambiguity in mean returns leads the ambiguity-averse investor to apply a larger tilt toward the SGMV portfolio. In line with this theoretical relationship, our manuscript proposes a method to establish the degree of ambiguity in mean returns that provides a robust out-of-sample performance.

The proposed robust shrinkage portfolio shares elements with the robust portfolio optimization literature. Goldfarb and Iyengar (2003) show that constructing the portfolio that is optimal under the worst-case scenario is a powerful technique "to combat the sensitivity of the optimal portfolio to statistical errors." Similarly, we show that constructing portfolio combinations that maximize our proposed robustness measure is equivalent to solving a robust optimization problem where the investor maximizes the worst-case scenario of the unknown OOSU mean. Therefore, our proposed robust shrinkage portfolio implicitly accounts for statistical errors affecting the estimation of the OOSU mean.

The theoretical link between the SDF and the returns of mean-variance portfolios allows us to relate our theory with the cross-sectional asset pricing literature that exploits a linear combination of test assets to construct an SDF model (Cochrane, 2005; Kozak, Nagel, and Santosh, 2020). In particular, we show in Section IA.2 of the Internet Appendix that our measure of OOSU is related to the out-of-sample fit of SDF models. This link allows us to apply our theory to demonstrate that the out-of-sample fit of models that exploit near-arbitrage opportunities in the construction of the SDF loadings are unreliable. Our theoretical results complement and substantiate the empirical findings of Kozak et al. (2018), who claim that *"in-sample [near-arbitrage opportunities] do not appear to reliably persist out of sample"*.

Finally, our work also speaks to the debate on the replicability of finance research (Harvey, Liu, and Zhu, 2016). In this debate, academics try to find out-of-sample evidence that confirms the validity of cross-sectional anomalies (Jensen, Kelly, and Pedersen, 2021). Similar to this line of research, our work is motivated by the need to create evaluation measures that assess the out-of-sample robustness of quantitative strategies. Our work addresses this issue by providing a theory to measure the out-of-sample performance risk of quantitative strategies that suffer from parameter uncertainty.

3 Mean-variance portfolios

In this section, we review the portfolio framework introduced by Markowitz (1952) where the investor has mean-variance preferences and knows the true distributional properties of stock returns. In particular, we assume that the N stock returns in excess of the risk-free rate have a vector of means μ and a positive-definite covariance matrix $\mathbf{\Sigma}$. In addition, we impose the standard constraint that the investor's wealth is fully allocated to the N risky assets, i.e., $w^\top e = 1$, where e is the N-dimensional vector of ones and w is a vector of portfolio weights. Then, the optimal mean-variance portfolio is the solution to the following quadratic program

$$\max_{w: w^\top e = 1} \quad U(w) = w^\top \mu - \frac{\gamma}{2} w^\top \mathbf{\Sigma} w, \tag{1}$$

where $U(w)$ is the *utility* of portfolio w and $\gamma > 0$ is the investor's risk-aversion coefficient. The solution to problem (1) is

$$w^\star = w_g + \frac{1}{\gamma} w_z, \tag{2}$$

where w_g is the global minimum-variance (GMV) portfolio,

$$w_g = \mathbf{\Sigma}^{-1} e (e^\top \mathbf{\Sigma}^{-1} e)^{-1}, \tag{3}$$

and w_z is a zero-cost portfolio (i.e., $w_z^\top e = 0$) defined as

$$w_z = \mathbf{B}\mu, \quad \mathbf{B} = \mathbf{\Sigma}^{-1}(\mathbf{I} - e w_g^\top). \tag{4}$$

For notational simplicity, we define the mean return and variance of the GMV portfolio w_g, and the return variance of the zero-cost portfolio w_z, as

$$\mu_g = w_g^\top \mu = \mu^\top \mathbf{\Sigma}^{-1} e (e^\top \mathbf{\Sigma}^{-1} e)^{-1}, \tag{5}$$

$$\sigma_g^2 = w_g^\top \mathbf{\Sigma} w_g = (e^\top \mathbf{\Sigma}^{-1} e)^{-1}, \tag{6}$$

$$\psi^2 = w_z^\top \mathbf{\Sigma} w_z = \mu^\top \mathbf{\Sigma}^{-1} \mu - \mu_g^2/\sigma_g^2, \tag{7}$$

respectively. Note that the return variance of the zero-cost portfolio, $\psi^2 \geq 0$, is equal to the difference of squared Sharpe ratios of the tangency portfolio and the GMV portfolio. In Section 5.4, we show that ψ^2 determines the contribution of low-variance principal components to the maximum attainable Sharpe ratio, and therefore it indicates whether near-arbitrage opportunities are available to investors.

It is straightforward to show that the utility of the mean-variance portfolio $w^\star$ is

$$U(w^\star) = U(w_g) + \frac{\psi^2}{2\gamma}. \tag{8}$$

Because $\psi^2/(2\gamma)$ is always positive, the optimal mean-variance portfolio always delivers a higher in-sample utility than the GMV portfolio. However, the mean-variance portfolio's in-sample optimality does not hold out of sample because of the estimation errors affecting the inputs of the portfolio problem. In particular, the impact of estimation errors in the vector of means on portfolio performance can be severe, as documented in prior literature (Merton, 1980; Chopra and Ziemba, 1993). Therefore, it is essential to account for parameter uncertainty in the construction of investment strategies, which we address in the next section.

4 The distribution of out-of-sample utility

The out-of-sample performance of estimated portfolios is uncertain due to estimation errors, as we note at the end of the previous section. This section sheds new light on the stochastic nature of portfolio performance by characterizing the asymptotic distribution of out-of-sample utility of sample portfolios. Section 4.1 lays out the main theoretical assumptions. Section 4.2 provides our definition of out-of-sample performance of sample portfolios, and Section 4.3 derives the asymptotic distribution of the out-of-sample performance of sample portfolios.

4.1 Theoretical assumptions

Let us consider the time $T+1$ portfolio return $w^\top r_{T+1}$, where r_{T+1} is the N-dimensional vector of stock returns in excess of the risk-free rate with mean μ and positive-definite covariance matrix $\mathbf{\Sigma}$. Using historical return data over the past T months $(r_1, \ldots, r_T)$, the investor estimates the vector of means μ and covariance matrix of stock returns $\mathbf{\Sigma}$ with their sample counterparts:

$$\hat{\mu} = \frac{1}{T}\sum_{t=1}^{T} r_t, \quad \widehat{\mathbf{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T}(r_t - \hat{\mu})(r_t - \hat{\mu})^\top. \tag{9}$$

Consistent with prior literature, we make the following two assumptions.

Assumption 1. *There are at least two stocks, $N \geq 2$, and the sample size is $T > N + 7$.*

Assumption 2. *The vector of stock returns at time t, r_t, follows a multivariate Gaussian distribution with vector of means μ and covariance matrix $\mathbf{\Sigma}$, and all return observations are independent and identically distributed (iid) over time.*

The condition $T > N + 7$ in Assumption 1 is needed to ensure that the out-of-sample utility variance derived in Section 5 exists. Assumption 2 is a standard assumption in the literature used for analytical tractability (Kan and Zhou, 2007; Ao et al., 2019). Under Assumption 2, $\hat{\mu}$ and $\widehat{\mathbf{\Sigma}}$ are independent and follow a multivariate Gaussian distribution and Wishart distribution, respectively.

While it is unlikely that the empirical data follow a Gaussian distribution, there are several reasons why Assumption 2 does not compromise the performance of the portfolio strategies that rely on this assumption. First, even when stock returns are non-Gaussian, there is a close relationship between expected utility and the mean-variance framework (Kroll, Levy, and Markowitz, 1984; Markowitz, 2014). Second, the economic losses of optimal portfolios that ignore fat tails in the distribution of stock returns are small, as demonstrated by Tu and Zhou (2004). Third, our empirical results show that the shrinkage portfolios calibrated under the assumption of Gaussian returns deliver good out-of-sample performance even for datasets with empirical return data where returns are likely not Gaussian.

4.2 Sample portfolios and out-of-sample utility

In practice, investors do not know the true vector of means and covariance matrix of stock returns, and instead, they estimate these parameters from historical return data using the sample estimates provided in Equation (9). Accordingly, the sample estimate of the mean-variance portfolio in (2), hereafter the SMV portfolio, is

$$\hat{w}^{\star} = \hat{w}_g + \frac{1}{\gamma}\hat{w}_z, \tag{10}$$

where $\hat{w}_g$ is the sample GMV portfolio, hereafter the SGMV portfolio, which is a function of $\widehat{\mathbf{\Sigma}}$ alone, and $\hat{w}_z$ is the sample zero-cost portfolio, which is a function of both $\hat{\mu}$ and $\widehat{\mathbf{\Sigma}}$.

The estimation risk affecting the SMV portfolio leads to suboptimal performance as noted by DeMiguel et al. (2009b). To combat the impact of parameter uncertainty, we consider shrinkage techniques, which help mitigate the impact of statistical errors on the performance of mean-variance portfolios. Indeed, Kan et al. (2021b) show that one can improve the *average* out-of-sample performance by *combining* the SMV portfolio $\hat{w}^{\star}$ with the SGMV portfolio $\hat{w}_g$. In Appendix IA.1, we show that this combination is also economically sound because it has a direct connection with the ambiguity-averse portfolios of Garlappi et al. (2007). Similarly, we consider a linear combination between the SMV and SGMV portfolios determined by the shrinkage intensity $\kappa \in [0,1]$:

$$\hat{w}^\star(\kappa) = (1-\kappa)\hat{w}_g + \kappa \hat{w}^\star \quad \text{with} \quad \kappa \in [0,1]. \tag{11}$$

Note that the shrinkage portfolio (11) contains as special cases the SMV and SGMV portfolios for $\kappa = 1$ and $\kappa = 0$, respectively. For this reason, the theory we develop in this manuscript for the shrinkage portfolio (11) is applicable to the SMV and SGMV portfolios.[6]

To evaluate the performance of $\hat{w}^\star(\kappa)$ while accounting for estimation risk, we follow Kan and Zhou (2007) and define the *out-of-sample utility* (OOSU) of an estimated portfolio $\hat{w}^\star(\kappa)$ as

$$U(\hat{w}^\star(\kappa)) = \hat{w}^\star(\kappa)^\top \mu - \frac{\gamma}{2}\hat{w}^\star(\kappa)^\top \boldsymbol{\Sigma} \hat{w}^\star(\kappa). \tag{12}$$

Note that because $\hat{w}^\star(\kappa)$ is estimated from a random sample, the OOSU is on its own a random variable, whose distribution we characterize in the following section.

4.3 Asymptotic distribution of out-of-sample utility

In this section, we characterize the asymptotic distribution of the OOSU for the shrinkage portfolio defined in Equation (11). The asymptotic distribution is derived as both the sample size T and the number of assets N go to infinity but their ratio converges to a constant, similar to the analysis in Ledoit and Wolf (2017), Ao et al. (2019), and Kan et al. (2021a). This result contains, as a particular case, the asymptotic distribution of the OOSU for the SMV and the SGMV portfolios when the shrinkage intensity is one and zero, respectively.[7]

Proposition 1. *Let $N, T \to \infty$, $N/T \to \rho \in [0,1)$, and Assumption 2 hold. Then, the out-of-sample utility of the shrinkage portfolio $\hat{w}^\star(\kappa)$ is asymptotically normal,*

$$\sqrt{T}\Big(U(\hat{w}^\star(\kappa)) - u(\kappa,\rho)\Big) \xrightarrow{d} \mathcal{N}(0, v(\kappa,\rho)), \tag{13}$$

[6] In Appendix IA.4.5, we extend our analysis to considering a shrinkage portfolio that combines the SMV portfolio, the SGMV portfolio, and the equally weighted portfolio as in Tu and Zhou (2011).

[7] All proofs are available in the Internet Appendix.

where the asymptotic mean is

$$u(\kappa,\rho) = \mu_g - \frac{\gamma}{2}\frac{\sigma_g^2}{1-\rho} + \frac{1}{\gamma}\frac{1}{1-\rho}\left(\kappa\psi^2 - \frac{\kappa^2}{2}\frac{\psi^2+\rho}{(1-\rho)^2}\right),$$

and the asymptotic variance is

$$v(\kappa,\rho) = v(0,\rho) + a_1(\rho)\kappa^4 + a_2(\rho)\kappa^3 + a_3(\rho)\kappa^2 + a_4(\rho)\kappa$$

with

$$\begin{aligned}
v(0,\rho) &= \frac{\sigma_g^2\psi^2}{1-\rho} + \frac{\gamma^2\sigma_g^4\rho}{2(1-\rho)^3},\\
a_1(\rho) &= \frac{(\psi^2+\rho/2)(\rho^2+3\rho+1)+\psi^4(2+\rho/2)}{\gamma^2(1-\rho)^7},\\
a_2(\rho) &= -\frac{2\psi^2(2\psi^2+\rho+1)}{\gamma^2(1-\rho)^5},\\
a_3(\rho) &= \frac{\psi^2(4\psi^2+1)}{\gamma^2(1-\rho)^3} + \frac{\sigma_g^2(1+\rho)(\psi^2+\rho)}{(1-\rho)^5},\\
a_4(\rho) &= -\frac{2\sigma_g^2\psi^2}{(1-\rho)^3}.
\end{aligned}$$

Proposition 1 shows that the OOSU of sample portfolios follows asymptotically a Gaussian distribution. One of the important implications of this result is that the entire distribution can be defined with the first two moments. For instance, as N and T increase with $N/T = \rho \in [0,1)$, the α-Value-at-Risk (e.g., $\alpha = 5\%$) of OOSU converges to

$$u(\kappa,\rho) - \lambda_{1-\alpha}\sqrt{v(\kappa,\rho)/T}, \tag{14}$$

where $\lambda_{1-\alpha}$ is the $(1-\alpha)$th percentile of a standard normal distribution. This corresponds to the portfolio robustness measure we propose in Section 6.

The asymptotic analysis presented in this section is interesting in its own right because it gives a complete description of the stochastic nature of out-of-sample performance in high-dimensional settings. Investors can easily exploit the results presented in this section to assess the performance uncertainty of sample portfolios. In empirical applications, both the cross-section and the time-series dimensions are finite. Therefore, in the following section, we study the finite-sample properties of out-of-sample performance.

5 Finite-sample out-of-sample utility risk

This section studies the OOSU mean and variance of portfolios estimated from finite samples. In particular, we derive the closed-form analytical expression of the finite-sample OOSU mean and variance for the shrinkage portfolio defined in Equation (11), which contains as particular cases the finite-sample OOSU mean and variance of the individual SMV and SGMV portfolios. We then discuss the monotonicity properties of OOSU variance and shed light on the connection between near-arbitrage opportunities and OOSU risk.

5.1 Out-of-sample utility mean

In the following proposition, we review some of the main results of Kan et al. (2021b) concerning the OOSU mean of the shrinkage portfolio $\hat{w}^\star(\kappa)$ in Equation (11).

Proposition 2 (Kan et al. (2021b))**.** *Let Assumptions 1 and 2 hold. Then,*

1. *The out-of-sample utility mean of the sample GMV portfolio is*

$$\mathbb{E}[U(\hat{w}_g)] = \mu_g - \frac{\gamma}{2}\frac{T-2}{T-N-1}\sigma_g^2. \tag{15}$$

2. *The out-of-sample utility mean of the shrinkage portfolio $\hat{w}^\star(\kappa)$ is*

$$\begin{aligned}\mathbb{E}[U(\hat{w}^\star(\kappa))] &= \mathbb{E}[U(\hat{w}_g)]\\ &+ \frac{1}{\gamma}\frac{T}{T-N-1}\left(\kappa\psi^2 - \kappa^2\left(\psi^2 + \frac{N-1}{T}\right)\frac{T(T-2)}{2(T-N)(T-N-3)}\right).\end{aligned} \tag{16}$$

3. *The shrinkage intensity $\kappa_E^\star$ maximizing out-of-sample utility mean in (16) is*

$$\kappa_E^\star = \frac{(T-N)(T-N-3)}{T(T-2)}\frac{\psi^2}{\psi^2 + \frac{N-1}{T}} \in [0,1]. \tag{17}$$

Finally, $\kappa_E^\star \to 1$ as $T \to \infty$. Thus, $\hat{w}^\star(\kappa_E^\star)$ is a consistent estimator of $w^\star$.

Note that the optimal shrinkage intensity $\kappa_E^\star$ in Proposition 2 is an oracle estimator that depends on the unknown parameter ψ^2. Kan et al. (2021b) rely on a feasible estimator of $\kappa_E^\star$ using the estimator of ψ^2 proposed by Kan and Zhou (2007) and find that the resulting portfolio delivers better out-of-sample performance than a wide range of benchmarks.

In the next section, we extend the analysis of Proposition 2 to study the OOSU variance of sample portfolios, and we utilize this result to construct robust shrinkage portfolios that balance OOSU mean and volatility in Section 6, where show that the robust portfolios deliver a consistently better out-of-sample performance than those that ignore OOSU risk.

5.2 Out-of-sample utility variance

We first define in the following lemma the OOSU variance of any random portfolio, which we use to obtain the OOSU variance of the shrinkage portfolio defined in Equation (11).

Lemma 1. *The out-of-sample utility variance of a random vector of portfolio weights $\hat{w}$ is*

$$\mathbb{V}[U(\hat{w})] = \mathbb{V}\Big[\hat{w}^\top \mu\Big] + \frac{\gamma^2}{4}\mathbb{V}\Big[\hat{w}^\top \mathbf{\Sigma}\hat{w}\Big] - \gamma\,\mathbb{C}\mathrm{ov}\Big[\hat{w}^\top\mu, \hat{w}^\top\mathbf{\Sigma}\hat{w}\Big]. \tag{18}$$

In the following proposition, we apply Lemma 1 to derive the closed-form analytical expression of the OOSU variance of the shrinkage portfolio defined in Equation (11).[8]

Proposition 3. *Let Assumptions 1 and 2 hold. Then, the out-of-sample utility variance of the shrinkage portfolio $\hat{w}^\star(\kappa)$ is*

$$\mathbb{V}[U(\hat{w}^\star(\kappa))] = \mathbb{V}[U(\hat{w}_g)] + \Delta(\kappa), \tag{19}$$

where

$$\mathbb{V}[U(\hat{w}_g)] = \frac{\sigma_g^2\psi^2}{T-N-1} + \frac{\gamma^2\sigma_g^4(N-1)(T-2)}{2(T-N-1)^2(T-N-3)} \tag{20}$$

is the out-of-sample utility variance of the sample GMV portfolio and $\Delta(\kappa)$ is a fourth-degree polynomial in κ,

$$\Delta(\kappa) = a_1\kappa^4 + a_2\kappa^3 + a_3\kappa^2 + a_4\kappa, \tag{21}$$

[8] We are thankful to Raymond Kan for his helpful feedback, which helped us obtain our main result in this section.

with the coefficients (a_1, a_2, a_3, a_4) *being functions of* γ, T, N, σ_g^2, *and* ψ^2*:*

$$
\begin{aligned}
a_1 =& \frac{1}{2\gamma^2} \frac{T^2(T-2)C(T,N,\psi^2)}{(T-N)^2(T-N-1)^2(T-N-2)(T-N-3)^2(T-N-5)(T-N-7)}, && (22)\\
a_2 =& -\frac{2\psi^2}{\gamma^2} \frac{T^2(T-2)(T+N-3+2T\psi^2)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)}, && (23)\\
a_3 =& \frac{\psi^2}{\gamma^2} \frac{2T(N+1)+T^2(T-N-3+2(T-N)\psi^2)}{(T-N)(T-N-1)^2(T-N-3)} \\
&+ \sigma_g^2 \frac{T(T-2)(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)}, && (24)\\
a_4 =& -2\sigma_g^2\psi^2 \frac{T(T-2)}{(T-N-1)^2(T-N-3)}, && (25)
\end{aligned}
$$

where

$$
\begin{aligned}
C(T,N,\psi^2) = & (2T\psi^2+N-1)(N^4+N^3T-3N^3-4N^2T^2+22N^2T-31N^2+NT^3 \\
& -7NT^2+13NT-5N+T^4-12T^3+53T^2-100T+70)+T^2\psi^4(N^3 \\
& +2N^2T-6N^2-7NT^2+40NT-53N+4T^3-34T^2+88T-70).
\end{aligned}
$$

Note from Proposition 3 that the OOSU variance of the shrinkage portfolio only depends on six parameters: the shrinkage intensity κ, the investor's risk-aversion coefficient γ, the number of stocks N, the sample size T, the return variance of the GMV portfolio σ_g^2, and the return variance of the zero-cost portfolio ψ^2. We now use the analytical expression of OOSU variance in Proposition 3 to obtain the following Corollary.

Corollary 1. *Provided that* ψ^2 *is strictly positive, there is a nonzero shrinkage intensity* $\kappa \in (0,1)$ *for which the shrinkage portfolio* $\hat{w}^\star(\kappa)$ *delivers a lower out-of-sample utility variance than that of the SMV and SGMV portfolios.*

Corollary 1 demonstrates that neither the SMV portfolio nor the SGMV portfolio delivers, individually, the lowest OOSU variance, and it is optimal to combine them to minimize OOSU variance in (19). We illustrate this point in Figure 1, where we depict in the horizontal axis the OOSU standard deviation of different shrinkage portfolios using the closed-form expression obtained in Proposition 3. We see that for the case considered in Figure 1, the shrinkage intensity that minimizes OOSU variance is $\kappa_V^\star = 0.0147 > 0$.

5.3 Monotonicity properties of out-of-sample utility variance

We now study the monotonicity properties of the OOSU variance of the shrinkage portfolio in (11), which we highlight in the following proposition.

Proposition 4. *The out-of-sample utility variance of the shrinkage portfolio* $\hat{w}^\star(\kappa)$

1. *decreases with the sample size* T *and converges to zero as* $T \to \infty$*,*

2. *increases with the number of stocks* N*, the return variance of the GMV portfolio* σ_g^2*, the return variance of the zero-cost portfolio* ψ^2*, and the shrinkage intensity* κ *if* $\kappa \geq \kappa_E^\star$*.*[9]

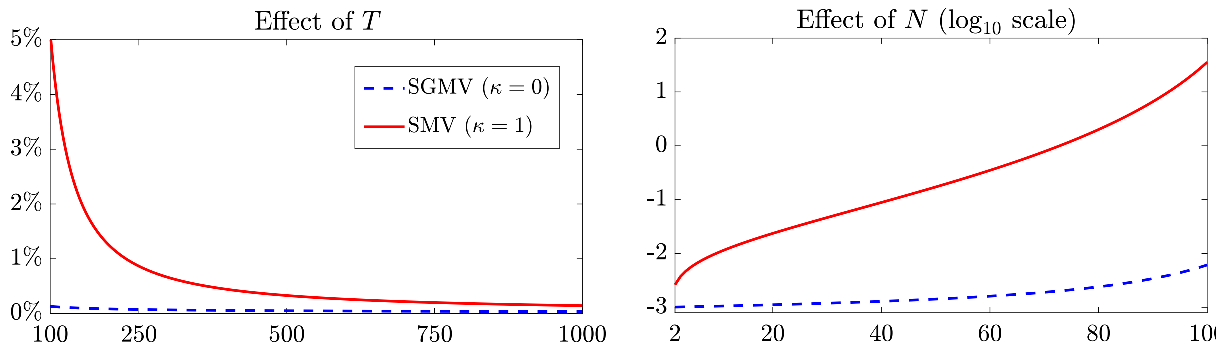
Proposition 4 provides the intuitive result that the OOSU risk of sample portfolios is substantial in high-dimensional settings where the number of assets N is large relative to the sample size T. Moreover, OOSU volatility increases with parameters σ_g^2 and ψ^2, which are the return variances of the GMV portfolio w_g and the zero-cost portfolio w_z, respectively. Finally, Proposition 4 shows that for a shrinkage intensity $\kappa \geq \kappa_E^\star$, the substantial exposure to the SMV portfolio leads to an increasing OOSU volatility as we increase κ. Therefore, κ needs to be smaller than $\kappa_E^\star$ in order to reduce OOSU volatility.

Figure 2 illustrates the results in Proposition 4. For conciseness, we only show the results for the sample size T and the number of stocks N. We calibrate the distributional parameters using the sample moments of the 25SBTM dataset. This gives a value of $\sigma_g = 0.0437$ and a value of $\psi^2 = 0.0625$. In addition, we set a risk-aversion coefficient of $\gamma = 3$. In the left Panel, we vary T while keeping a fixed $N = 25$. In the right Panel, we vary N while keeping a fixed $T = 120$. We study the OOSU volatility of the SGMV portfolio and the SMV portfolio.

The left Panel in Figure 2 shows that the OOSU standard deviation of the SMV portfolio is substantially larger than that of the SGMV portfolio. Specifically, for a realistic sample size of $T = 120$ monthly observations, the monthly OOSU standard deviation of the SMV portfolio is approximately 3.34%, which is 29 times larger than that of the SGMV portfolio. Additionally, consistent with Proposition 4, we observe that the OOSU standard deviation decreases with the sample size. However, it is worth noting that the SMV portfolio requires

[9] This is a sufficient but not necessary condition.

Figure 2: Effect of sample size and number of stocks on out-of-sample utility volatility



Notes. This figure depicts the out-of-sample utility standard deviation of the SGMV portfolio ($\kappa = 0$, dashed blue line) and the SMV portfolio ($\kappa = 1$, solid red line). The population vector of means and covariance matrix of stock excess returns are calibrated from the monthly return data of the 25 portfolios of stocks sorted on size and book-to-market. We set a risk-aversion coefficient of $\gamma = 3$. We vary the sample size T in the left panel while keeping a fixed $N = 25$, and we vary the number of stocks N in the right panel while keeping a fixed $T = 120$. The values in the right panel are in log-scale for visibility.

an unrealistically large sample size of T = 13,340 monthly observations to have a smaller OOSU volatility than the SGMV portfolio.

The right Panel in Figure 2 shows that OOSU standard deviation increases with the number of stocks N as demonstrated in Proposition 4. The effect of the number of stocks N is particularly severe for the SMV portfolio, for which we see that OOSU volatility increases much more rapidly than for the SGMV portfolio.

5.4 Relation with near-arbitrage opportunities

In Section 5.3, we derive the monotonicity properties of OOSU variance for any combination of the SMV and SGMV portfolios. We now use these results to discuss the connection between OOSU risk and near-arbitrage opportunities. First, we introduce the following assumption for tractability, which is also employed by Kozak et al. (2018).

Assumption 3. *Consider the eigenvalue decomposition of the covariance matrix of stock returns, $\mathbf{\Sigma} = \mathbf{V}\mathbf{D}\mathbf{V}^\top$, where $\mathbf{D} = \mathrm{diag}(d_1, \dots, d_N)$ is the diagonal matrix of eigenvalues, and $\mathbf{V} = [v_1, \dots, v_N]$ is the matrix of eigenvectors. We assume that the first eigenvector is proportional to the equally weighted portfolio, that is, $v_1 = e/\sqrt{N}$.*

Assumption 3 indicates that the main driver of stock-return variation, i.e., the first eigenvector of the return covariance matrix, is proportional to the equally weighted portfolio. This is a mild assumption because, in unreported results, we show that empirically the first principal component of stock returns is highly correlated with the returns of the equally weighted portfolio, with a correlation of nearly 100% across the six datasets we consider in our analysis.

The following proposition employs Assumption 3 to decompose the return variance of the zero-cost portfolio, ψ^2 in (7), as the sum of squared Sharpe ratios of low-variance principal components.

Proposition 5. *Let Assumption 3 hold. Then, the return variance of the zero-cost portfolio is*

$$\psi^2 = \sum_{i>1}^{N} SR_{PC_i}^2, \tag{26}$$

where $SR_{PC_i} = v_i^\top \mu/\sqrt{d_i}$ is the Sharpe ratio of the ith principal component of stock returns.

Proposition 5 reveals that ψ^2 is determined by the squared Sharpe ratios of low-variance principal components, and thus, a large value of ψ^2 indicates the presence of *near-arbitrage opportunities*. Kozak et al. (2018) document that if there are low-variance principal components with substantial contribution to the overall performance of the tangency portfolio, these near-arbitrage opportunities are hard to exploit out of sample. Moreover, Kozak et al. (2020, p.278) cast doubt on the existence of large values of ψ^2 because they argue that "[it is] implausible that a principal component with low eigenvalue could contribute substantially to the volatility of the SDF and hence to the overall maximum squared Sharpe ratio." Our theoretical result in Proposition 4 that the OOSU variance of sample portfolios increases with ψ^2 complements and substantiates these claims. Specifically, even though our results do not suggest that it is implausible to find near-arbitrage opportunities, we show they are risky because they imply higher out-of-sample performance volatility.

Finally, from an investment perspective it is interesting to understand whether quantitative strategies exploiting *in-sample* near-arbitrage opportunities are subject to higher OOSU risk. The following proposition addresses this concern.

Proposition 6. *Let Assumptions 1 and 2 hold. Then, provided that $T > N+9$, the covariance between $\hat{\psi}^2 = \hat{\mu}^\top \hat{\boldsymbol{\Sigma}}^{-1}\hat{\mu} - (\hat{\mu}_g/\hat{\sigma}_g)^2$ and $(U(\hat{w}(\kappa)) - \mathbb{E}[U(\hat{w}(\kappa))])^2$ exists and is always positive.*

Proposition 6 demonstrates that sample portfolios exploiting in-sample near-arbitrage opportunities will face larger OOSU risk because the in-sample measure of near-arbitrage opportunities, $\hat{\psi}^2$, covaries positively with the squared OOSU deviations from the OOSU mean of the shrinkage portfolio that combines the SMV and SGMV portfolios.

6 A new portfolio robustness measure

We now use the results in Section 5 to propose a new portfolio robustness measure defined as the difference between OOSU mean and a multiple of OOSU standard deviation. For notational simplicity, our presentation focuses on the more general shrinkage portfolio that combines the SMV and the SGMV portfolios. Section 6.1 introduces the robustness measure. Section 6.2 studies the shrinkage portfolio that optimizes the proposed robustness measure. Section 6.3 explains how we estimate the shrinkage intensities. Section 6.4 describes the monotonicity properties of the robustness measure. Finally, Section 6.5 relates our proposed metric to the literature on robust optimization.

6.1 A new robustness measure

In our view, a robust portfolio should deliver good average performance *and* also a stable performance. In line with this view, we define the portfolio robustness measure for a given estimated portfolio $\hat{w}$ as the difference between OOSU mean and a multiple of OOSU standard deviation:

$$R(\hat{w}) = \mathbb{E}[U(\hat{w})] - \lambda\sqrt{\mathbb{V}[U(\hat{w})]}, \tag{27}$$

where $\lambda \geq 0$ determines the weight that OOSU risk has on our robustness measure. Note that for $\lambda = 0$, we recover the OOSU mean criterion proposed by Kan and Zhou (2007). We dub our robustness measure the *mean-risk OOSU*. Our proposed mean-risk OOSU measure captures our view of portfolio robustness, and maximizing this metric delivers an efficient tradeoff between OOSU mean and standard deviation. In addition, the result in Proposition 1 demonstrates that the robustness measure we propose in this section converges to the $F(-\lambda)$-Value-at-Risk of the OOSU distribution of the shrinkage portfolio (11) when N and T go to infinity, and the ratio N/T converges to a value in the interval [0,1).[10] Therefore, maximizing our proposed robustness measure minimizes asymptotically the left-tail risk in the out-of-sample utility of estimated portfolios.

6.2 The robust optimal portfolio

We now define our robust shrinkage portfolio that combines the SMV and SGMV portfolios to maximize the mean-risk OOSU measure defined in Section 6.1. Formally, the intensity of the robust shrinkage portfolio is the solution to the following problem:

$$\kappa_R^\star = \arg\max_{\kappa\in[0,1]} R(\hat{w}^\star(\kappa)). \tag{28}$$

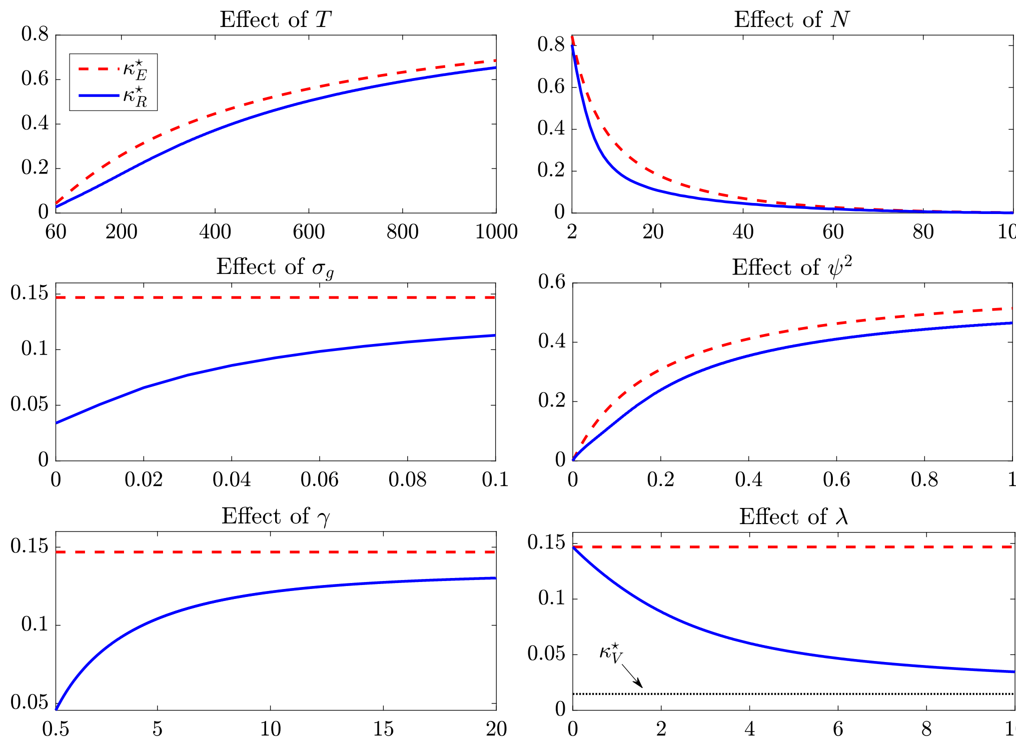
Problem (28) can be easily solved numerically using the analytical expressions for the OOSU mean in (16) and for the OOSU variance in (19). Note that we recover $\kappa_R^\star = \kappa_E^\star$ when $\lambda = 0$ and $\kappa_R^\star = \kappa_V^\star$ when $\lambda \to \infty$, where $\kappa_V^\star$ is the shrinkage intensity minimizing OOSU variance.

Our measure of portfolio robustness resembles the efficient frontier of Markowitz (1952). In our case, instead of obtaining the combination of stocks that achieves the optimal tradeoff between mean return and variance, we obtain the combination of estimated portfolios that achieves the optimal tradeoff between OOSU mean and variance. Figure 1 depicts the OOSU efficient frontier for the 25SBTM dataset, $T = 120$, and $\gamma = 3$. First, note that the shrinkage intensity $\kappa_E^\star$ proposed by Kan et al. (2021b) delivers the portfolio on the OOSU efficient frontier that maximizes OOSU mean. Second, we observe that the shrinkage portfolio that maximizes the mean-risk OOSU metric with $\lambda = 2$ delivers an OOSU standard deviation 21% lower than that of the shrinkage portfolio maximizing OOSU mean. To achieve this substantial reduction in OOSU risk, the shrinkage portfolio that exploits $\kappa_R^\star$ only sacrifices a small OOSU mean of 5.3% relative to the shrinkage portfolio that exploits $\kappa_E^\star$. Accordingly, our proposed robust shrinkage approach can deliver portfolios with a stable out-of-sample performance that perform well on average.

In the following proposition, we formally prove two important properties of the shrinkage intensity $\kappa_R^\star$.

[10] $F(x)$ is the cumulative distribution function of the standardized OOSU of the shrinkage portfolio.

Figure 3: Monotonicity properties of optimal shrinkage intensities



Notes. This figure depicts the shrinkage intensity $\kappa_E^\star$ maximizing out-of-sample utility mean (dotted red line), and the shrinkage intensity $\kappa_R^\star$ maximizing the portfolio robustness measure in (27) (solid blue line), for different values of the six parameters that influence the portfolio robustness measure introduced in Section 6.1. The population vector of means and covariance matrix of excess returns are calibrated from the monthly return data of the 25 portfolios of stocks sorted on size and book-to-market. The base-case values of the six parameters are $T = 120$, $N = 25$, $\sigma_g = 0.0437$, $\psi^2 = 0.0625$, $\gamma = 3$, and $\lambda = 2$. In each plot, we change the value of one of these six parameters while keeping the other five equal to the base-case value. In the bottom-right plot, $\kappa_V^\star$ is the shrinkage intensity minimizing out-of-sample utility variance.

Proposition 7. *Let Assumptions 1 and 2 hold. Then, the shrinkage intensity $\kappa_R^\star$ solving (28) has the following properties:*

1. *$\kappa_R^\star \to 1$ as $T \to \infty$. Thus, $\hat{w}^\star(\kappa_R^\star)$ is a consistent estimator of $w^\star$.*

2. *$\kappa_V^\star \leq \kappa_R^\star \leq \kappa_E^\star$, where $\kappa_V^\star$ minimizes the out-of-sample utility variance and $\kappa_E^\star$ maximizes the out-of-sample utility mean of the shrinkage portfolio, respectively.*

The first result of Proposition 7 shows that our proposed robust shrinkage portfolio is asymptotically optimal. The second result of Proposition 7 demonstrates that while an investor facing parameter uncertainty can increase her average OOSU by shrinking the SMV portfolio toward the SGMV portfolio, a more substantial shrinkage toward the SGMV portfolio is needed to reduce OOSU variance further and enhance portfolio robustness. Using the insight in Corollary 1 that $\kappa_V^\star > 0$ if $\psi^2 > 0$, the second result of Proposition 7 also implies that neither the SMV portfolio nor the SGMV portfolio optimizes our proposed measure of portfolio robustness and, hence, it is necessary to combine them using intensity $\kappa_R^\star$ to achieve the maximal robust performance.

In Figure 3, we illustrate the monotonicity properties of the optimal shrinkage intensity $\kappa_R^\star$ that maximizes the mean-risk OOSU measure. We calibrate the parameters required to obtain the optimal shrinkage intensity using the 25SBTM dataset. In particular, we have $N = 25$, $\psi^2 = 0.0625$, and $\sigma_g = 0.0437$. In addition, we set $T = 120$, $\gamma = 3$, and $\lambda = 2$. We then change one parameter at a time to study the monotonicity properties of $\kappa_R^\star$.

We observe from Figure 3 that both $\kappa_E^\star$ and $\kappa_R^\star$ increase with T and decrease with N. This result is intuitive because statistical errors affecting the estimated moments of stock returns decrease with the ratio T/N and, thus, less shrinkage toward the SGMV portfolio is necessary when this ratio increases. Also, the difference between $\kappa_E^\star$ and $\kappa_R^\star$ becomes smaller as T increases because they both converge to one as T goes to infinity. Second, while $\kappa_E^\star$ is independent of σ_g, the proposed shrinkage intensity $\kappa_R^\star$ increases with σ_g because, as the return volatility of the GMV portfolio increases, shrinking toward the SGMV portfolio becomes less attractive in terms of OOSU risk. Third, we observe that both shrinkage intensities increase with ψ^2, but $\kappa_R^\star$ increases less rapidly because as shown in Proposition 4, the OOSU standard deviation of the shrinkage portfolio increases with ψ^2. Fourth, while $\kappa_E^\star$ is independent of γ, the proposed shrinkage intensity $\kappa_R^\star$ increases with γ and gets closer to $\kappa_E^\star$. This is because as γ increases, the exposure to sample mean returns decreases, and this has the effect of reducing OOSU standard deviation, which gives more relevance to the OOSU mean in the mean-risk OOSU measure. Fifth, as the coefficient λ increases, the OOSU standard deviation of the shrinkage portfolio becomes a more relevant element of the mean-risk OOSU criterion. Therefore, the shrinkage intensity $\kappa_R^\star$ converges to $\kappa_V^\star$ as λ increases. This insight is consistent with our second result in Proposition 7.

6.3 Estimation of optimal shrinkage intensities

The optimal shrinkage intensity $\kappa_R^\star$ maximizing the mean-risk OOSU of the shrinkage portfolio depends on the true moments of stock returns via σ_g^2 and ψ^2. Similarly, the shrinkage intensity $\kappa_E^\star$ maximizing OOSU mean depends on parameter ψ^2. Since these parameters are unknown, Section IA.3 of the Internet Appendix studies estimators of these parameters that are statistically consistent. Specifically, we estimate ψ^2 with the estimator of Kan and Zhou (2007). Moreover, we estimate σ_g^2 via the estimator of Frahm and Memmel (2010), which corresponds to the sample return variance of the shrinkage portfolio that combines the equally weighted and SGMV portfolios to minimize expected out-of-sample variance. In the rest of the manuscript, we denote the estimated intensities that exploit the consistent estimators of ψ^2 and σ_g^2 as $\hat{\kappa}_R^\star$ and $\hat{\kappa}_E^\star$.[11]

6.4 Monotonicity properties of the robustness measure

In the following proposition, we provide some monotonicity properties for the mean-risk OOSU measure of the shrinkage portfolio (11).

Proposition 8. *The mean-risk out-of-sample utility of the shrinkage portfolio $\hat{w}^\star(\kappa)$*

1. *increases with the sample size T and the mean return of the GMV portfolio μ_g,*
2. *decreases with the number of stocks N, the return variance of the GMV portfolio σ_g^2, and the shrinkage intensity κ if $\kappa \geq \kappa_E^\star$,*
3. *increases with ψ^2 at a smaller rate as λ increases, i.e., $\frac{\partial^2}{\partial\psi^2\partial\lambda} R(\hat{w}^\star(\kappa)) < 0$.*

Proposition 8 demonstrates that the mean-risk OOSU measure increases with T and decreases with N. This is a desirable property of our proposed robustness metric because increasing T and decreasing N reduces the statistical errors affecting the estimated moments of stock returns and their impact on sample portfolios. Moreover, the mean-risk OOSU measure increases with μ_g because the OOSU mean increases with μ_g and the OOSU standard deviation is independent of μ_g. On the contrary, the mean-risk OOSU decreases with σ_g^2 because the OOSU mean is decreasing in σ_g^2, and the OOSU standard deviation is increasing in σ_g^2 as shown in Proposition 4. Proposition 8 also demonstrates that allocating more weight

[11]The result in Part 2 of Proposition 7 holds for any value of σ_g^2 and ψ^2, hence the estimated shrinkage intensities also obey the inequality $\hat{\kappa}_R^\star \leq \hat{\kappa}_E^\star$. Moreover, they remain asymptotically optimal.

to the SMV portfolio than $\kappa_E^\star$ necessarily deteriorates the mean-risk OOSU measure, which is why $\kappa_R^\star \leq \kappa_E^\star$ as highlighted in Proposition 7.

Finally, Proposition 8 also shows that increasing the coefficient λ mitigates the impact that parameter ψ^2 has on the robustness measure. This is because while ψ^2 generally has a positive impact on OOSU mean,[12] it also positively impacts OOSU volatility. Therefore, the near-arbitrage opportunities captured by parameter ψ^2 do not represent a free-lunch to investors because while average out-of-sample performance may increase, it also becomes riskier. This is captured by our proposed mean-risk OOSU measure and explains why the robust shrinkage intensity $\kappa_R^\star$ does not increase with ψ^2 as fast as $\kappa_E^\star$ in Figure 3.

6.5 Relation to robust optimization

In this section, we interpret our proposed mean-risk OOSU criterion through the lenses of robust optimization.[13] Under this approach, the mean-variance investor is averse to the *ambiguity* around the true, *but unknown*, OOSU mean and maximizes the worst-case scenario assuming that the true OOSU mean lies within a bounded region. In particular, we assume that the true OOSU mean belongs to the following uncertainty set:

$$\mathcal{S}(\lambda, \kappa) = \left\{ x \in \left[\widehat{\mathbb{E}}[U(\hat{w}^\star(\kappa))] - \lambda\sqrt{\widehat{\mathbb{V}}[U(\hat{w}^\star(\kappa))]}, \widehat{\mathbb{E}}[U(\hat{w}^\star(\kappa))] + \lambda\sqrt{\widehat{\mathbb{V}}[U(\hat{w}^\star(\kappa))]} \right] \right\}, \quad (29)$$

where $\widehat{\mathbb{E}}[U(\hat{w}^\star(\kappa))]$ and $\widehat{\mathbb{V}}[U(\hat{w}^\star(\kappa))]$ are the estimated OOSU mean and variance of the shrinkage portfolio $\hat{w}^\star(\kappa)$. Note that in this case parameter $\lambda \geq 0$ determines the level of uncertainty around the OOSU mean. The uncertainty set in (29) can be interpreted as a confidence interval similar to Garlappi et al. (2007). Accordingly, an ambiguity-averse investor who wants to maximize OOSU mean solves the robust optimization problem

$$\max_{\kappa \in [0,1]} \min_{\mathcal{S}(\lambda,\kappa)} \mathbb{E}[U(\hat{w}^\star(\kappa))] = \max_{\kappa \in [0,1]} \widehat{\mathbb{E}}[U(\hat{w}^\star(\kappa))] - \lambda\sqrt{\widehat{\mathbb{V}}[U(\hat{w}^\star(\kappa))]}. \quad (30)$$

[12] Specifically, OOSU mean in (16) increases with ψ^2 if $\kappa \leq 2(T-N)(T-N-3)/(T(T-2))$.

[13] There is extensive literature on robust optimization and portfolio theory. One of the most prominent papers in this literature is Goldfarb and Iyengar (2003).

Problem (30) delivers our estimated robust shrinkage intensity, i.e., $\hat{\kappa}_R^\star$. Therefore, our methodology implicitly accounts for the estimation errors in OOSU mean, which can contaminate the estimated shrinkage intensity that maximizes OOSU mean and deteriorate portfolio performance (Kan and Wang, 2021). We confirm this finding in the simulation analysis of Section 7.2, where we find that our robust shrinkage portfolio often delivers a larger OOSU mean than that of the shrinkage portfolio that is designed to maximize OOSU mean.

7 Performance analysis

In this section, we characterize the economic benefits from exploiting our measure of portfolio robustness in the construction of quantitative strategies. In particular, we compare the performance of our robust shrinkage portfolio with that of several other benchmark portfolio strategies. In Section 7.1, we describe the data used in the performance analysis. In Section 7.2, we evaluate the performance of our strategy using simulated data and we empirically assess the relationship between OOSU risk and near-arbitrage opportunities. In Section 7.3, we evaluate the performance of our strategy using empirical data.

7.1 Data

We use the monthly excess returns of six datasets. The first four datasets are downloaded from Kenneth French's website: (i) 10 momentum portfolios (*10MOM*) from January 1927 through December 2019, (ii) 25 portfolios formed on size and book-to-market (*25SBTM*) from January 1927 through December 2019, (iii) 25 portfolios formed on operating profitability and investment (*25OPINV*) from July 1963 through December 2019, (iv) 49 industry portfolios (*49IND*) from July 1969 through December 2019. The last two datasets come from the 23 anomalies considered by Novy-Marx and Velikov (2016) and are downloaded from Robert Novy-Marx's website: (v) the long and short legs of eight low-turnover anomalies (*16LTANOM*) from July 1963 through December 2013 and (vi) the long and short legs of all the 23 anomalies (*46ANOM*) from July 1973 through December 2013.[14]

[14]We thank Kenneth French and Robert Novy-Marx for making their data publicly available.

7.2 Simulated returns

We use two different methods to simulate monthly return data. In the first method, we draw observations from a Gaussian distribution. In the second method, our return data is not Gaussian, and instead, we simulate data using the bootstrap method of Efron (1979). For the construction of the simulated data, we use the six datasets described in Section 7.1.

We now explain how we construct the Gaussian simulated returns. For each of the six empirical datasets, we compute the sample vector of means $\hat{\mu}$ and sample covariance matrix $\widehat{\mathbf{\Sigma}}$ employing the entire sample, and use these sample estimates as the population parameters of a multivariate Gaussian distribution $\mathcal{N}(\hat{\mu}, \widehat{\mathbf{\Sigma}})$ from which we draw T observations, where $T \in (120, 180, 240)$. We construct $M = 100{,}000$ simulated datasets of T observations using this method and compute the estimated shrinkage portfolio $\hat{w}_m(\hat{\kappa}_m)$ for each of the M simulated datasets. Then, the OOSU mean, OOSU variance, and mean-risk OOSU of the estimated shrinkage portfolio $\hat{w}^\star(\hat{\kappa})$ are approximated as

$$\mathbb{E}[U(\hat{w}^\star(\hat{\kappa}))] \approx \frac{1}{M}\sum_{m=1}^{M} U(\hat{w}^\star_m(\hat{\kappa}_m)), \tag{31}$$

$$\mathbb{V}[U(\hat{w}^\star(\hat{\kappa}))] \approx \frac{1}{M}\sum_{m=1}^{M} \left(U(\hat{w}^\star_m(\hat{\kappa}_m)) - \mathbb{E}[U(\hat{w}^\star(\kappa))]\right)^2, \tag{32}$$

$$R(\hat{w}^\star(\hat{\kappa})) \approx \mathbb{E}[U(\hat{w}^\star(\hat{\kappa}))] - \lambda\sqrt{\mathbb{V}[U(\hat{w}^\star(\hat{\kappa}))]}, \tag{33}$$

where $U(\hat{w}^\star_m(\hat{\kappa}_m))$ is the investor's out-of-sample utility defined as in Equation (12) of the estimated shrinkage portfolio $\hat{w}^\star_m(\hat{\kappa}_m)$ obtained from the mth simulated dataset. We set the risk-aversion coefficient to $\gamma = 3$ as in Kan and Zhou (2007) and Kan et al. (2021b) and we set the coefficient λ to $\lambda = 2$, and therefore the robustness measure we consider in the performance analysis is asymptotically equivalent to the 2.5%-Value-at-Risk of OOSU as shown in Section 4.3.[15]

The simulation with Gaussian data is interesting because the theoretical results rely on the assumption that stock returns are iid multivariate Gaussian; see Assumption 2. However, this assumption does not hold in practice, and therefore, the second type of simulated data

[15] In unreported results, we confirm that using different values of λ provides similar insights.

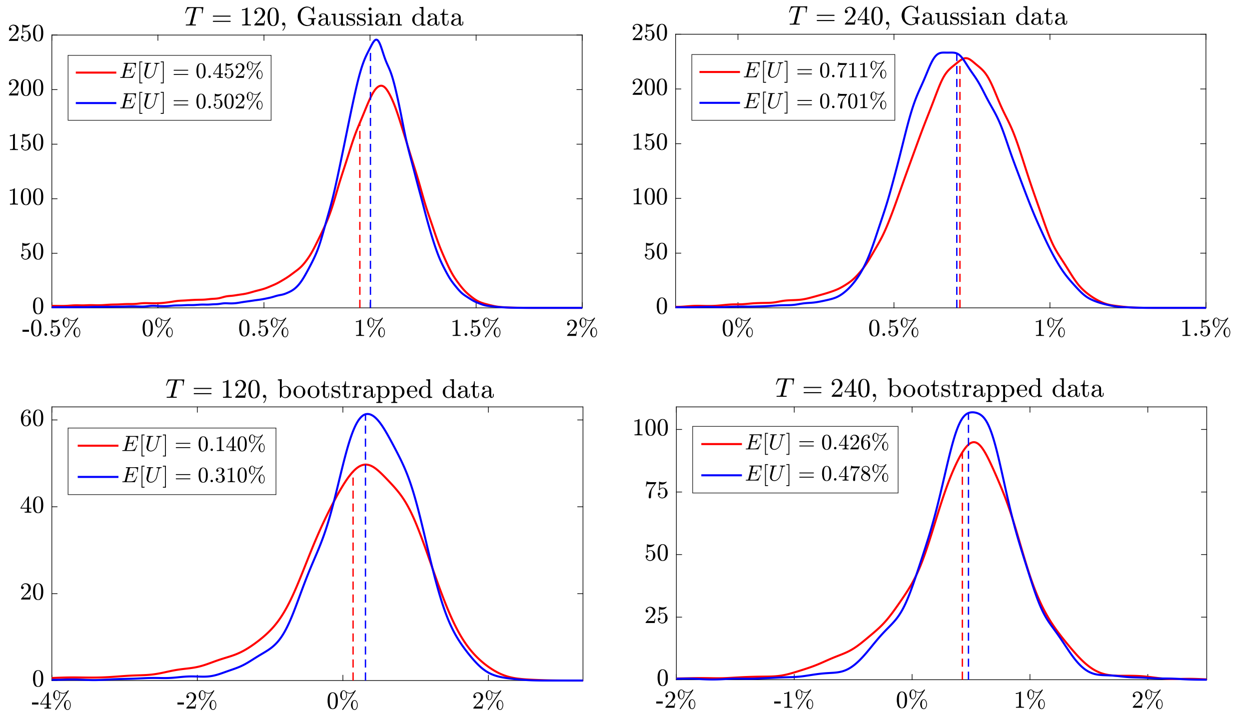
is obtained by bootstrapping return data from the original sample. In particular we create 1,000 bootstrap samples of $2T$ return observations, where $T \in (120, 180, 240)$. For each bootstrap sample of $2T$ observations, we use the first half to estimate the shrinkage portfolio and evaluate its performance in the second half of the sample. We compute the OOSU mean, variance, and the mean-risk OOSU as in Equations (31)–(33) from the 1,000 OOSU observations obtained from the 1,000 bootstrap samples.

Panel A of Table 1 reports the OOSU mean, standard deviation, and the mean-risk OOSU for the simulated Gaussian data. We consider the shrinkage portfolio with the estimated intensity $\hat{\kappa}_E^\star$ that maximizes OOSU mean and the shrinkage portfolio with the estimated intensity $\hat{\kappa}_R^\star$ that maximizes our proposed robustness measure with $\lambda = 2$. We observe that for a sample size of $T = 120$ observations, the shrinkage portfolio that exploits $\hat{\kappa}_R^\star$ delivers a larger OOSU mean than that of the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ for four datasets. This suggests that $\hat{\kappa}_R^\star$ emerges as a robust shrinkage intensity that is subject to lower estimation risk than $\hat{\kappa}_E^\star$, which allows our robust shrinkage portfolio to outperform in terms of OOSU mean the shrinkage portfolio designed to optimize it. Indeed, in unreported results, we find that the estimated $\hat{\kappa}_E^\star$ has a larger sampling variability than $\hat{\kappa}_R^\star$ and, hence, the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ is more sensitive to estimation errors. In addition, we observe that the shrinkage portfolio that exploits $\hat{\kappa}_E^\star$ delivers an OOSU that is notably more volatile than that of the shrinkage portfolio that exploits $\hat{\kappa}_R^\star$. The difference is particularly large for the case with a sample size of $T = 120$ observations. Accordingly, the shrinkage portfolio that exploits the intensity $\hat{\kappa}_E^\star$ delivers a smaller mean-risk OOSU than that obtained by the shrinkage portfolio that exploits $\hat{\kappa}_R^\star$.[16]

Panel B of Table 1 reports the performance of the two shrinkage portfolios in the bootstrap experiment. In terms of OOSU mean, we see that the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ performs remarkably better. Specifically, it outperforms the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ in five out of six datasets across all sample sizes. In addition, the shrinkage intensity $\hat{\kappa}_E^\star$ yields an OOSU volatility that is, on average across all datasets, 32%, 24%, and 21% larger than that of $\hat{\kappa}_R^\star$ for $T = 120$, 180, and 240 months, respectively. The results in this panel

[16]In unreported results, we show that the shrinkage portfolio constructed with $\hat{\kappa}_R^\star$ outperforms the SMV and SGMV portfolios in terms of OOSU mean, OOSU standard deviation, and mean-risk OOSU.

Figure 4: Density of monthly out-of-sample utility in simulated data



Notes. This figure depicts the monthly out-of-sample utility densities of the estimated shrinkage portfolios maximizing out-of-sample utility mean ($\hat{\kappa}^{\star}_E$, in red) and the portfolio robustness measure in Section 6 ($\hat{\kappa}^{\star}_R$, in blue). The top figures depict the density function of the out-of-sample utilities of the shrinkage portfolios from the 100,000 simulated samples of T observations drawn from a multivariate Gaussian distribution whose moments are calibrated from the dataset of monthly excess returns of 25 portfolios of stocks sorted on size and book-to-market. The bottom plots are obtained by bootstrapping (with replacement) 1,000 samples of $2T$ observations from the dataset of 25 portfolios of stocks sorted on size and book-to-market, where the first half of the bootstrap sample is used to estimate the two shrinkage portfolios and the second half is used to evaluate the out-of-sample utility of the shrinkage portfolios estimated in the first half of the sample. We consider a sample size of $T = 120$ and 240 monthly observations, a risk-aversion coefficient of $\gamma = 3$, and a coefficient $\lambda = 2$ for the portfolio robustness measure.

suggest that the performance of the shrinkage portfolio exploiting $\hat{\kappa}^{\star}_E$ deteriorates relative to that of the robust portfolio when the data is not Gaussian.

Figure 4 depicts the OOSU density of the estimated shrinkage portfolios for the simulated return data that uses the *25SBTM* dataset. The figure shows that, when the sample size is $T = 120$ or when the data is not Gaussian, the shrinkage portfolio that exploits $\hat{\kappa}^{\star}_R$ has a larger OOSU mean and a smaller OOSU volatility than the shrinkage portfolio exploiting $\hat{\kappa}^{\star}_E$. In addition, the OOSU density of the shrinkage portfolio exploiting $\hat{\kappa}^{\star}_E$ has a heavier left tail than that of the shrinkage portfolio exploiting $\hat{\kappa}^{\star}_R$. This result is consistent with the idea that our robust portfolio minimizes the tail risk of OOSU given the connection that we draw in Section 6.1 between our proposed robustness measure and the asymptotic Value-at-Risk of OOSU. The lower downside risk offered by our robust shrinkage portfolio represents an additional advantage of our proposed methodology.

Finally, we consider an extra source of portfolio robustness by shrinking the sample covariance matrix toward the identity as in Ledoit and Wolf (2004). In particular, we consider the following shrinkage covariance matrix:

$$\widehat{\mathbf{\Sigma}}_{\text{sh}} = (1-\delta)\widehat{\mathbf{\Sigma}} + \delta\bar{\sigma}^2\mathbf{I}, \tag{34}$$

where $\bar{\sigma}^2$ is the cross-sectional average of asset-return variances and δ is the shrinkage intensity. The eigenvalue decomposition of this shrinkage covariance matrix is

$$\widehat{\mathbf{\Sigma}}_{\text{sh}} = \widehat{\boldsymbol{V}}\Big((1-\delta)\widehat{\boldsymbol{D}} + \delta\bar{\sigma}^2\mathbf{I}\Big)\widehat{\boldsymbol{V}}^{\top}, \tag{35}$$

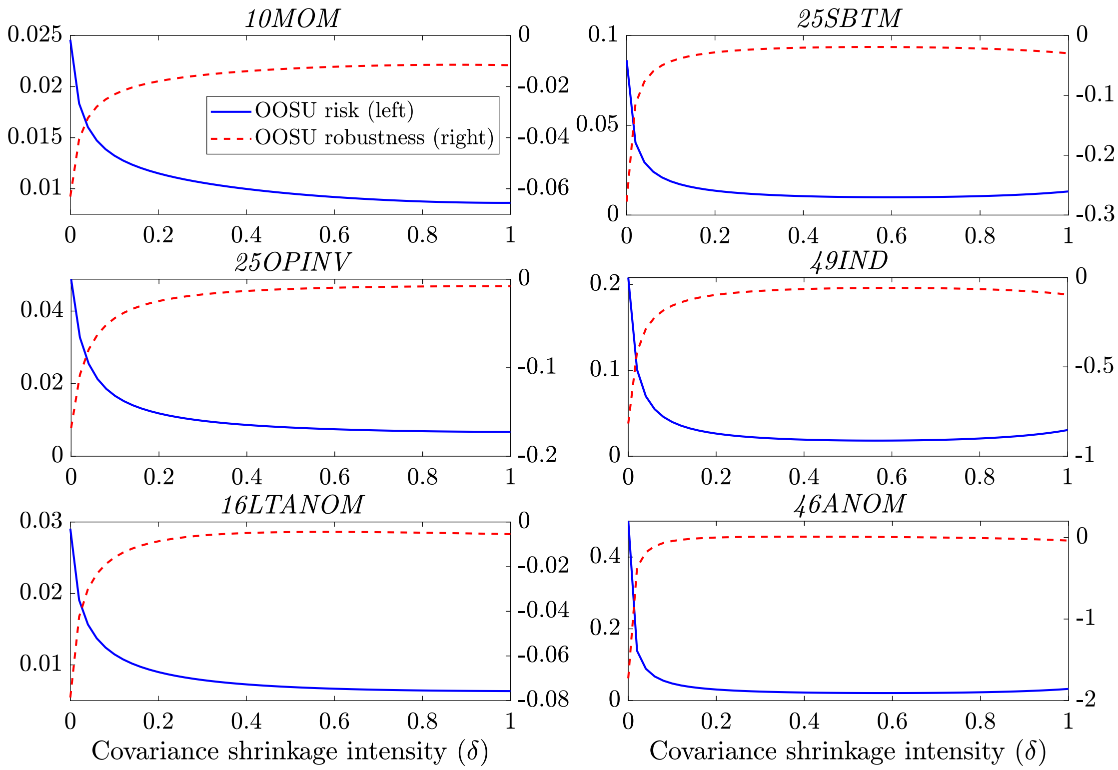
where $\widehat{\boldsymbol{V}}$ and $\widehat{\boldsymbol{D}}$ are the matrix of eigenvectors and the diagonal matrix of eigenvalues, respectively, obtained from $\widehat{\mathbf{\Sigma}}$. Taking the derivative of the eigenvalues $d_{\text{sh},i}$'s of the shrinkage covariance matrix with respect to the shrinkage intensity δ gives

$$\frac{\partial d_{\text{sh},i}}{\partial\delta} = \frac{\partial\Big((1-\delta)\hat{d}_i + \delta\bar{\sigma}^2\Big)}{\partial\delta} = \bar{\sigma}^2 - \hat{d}_i, \tag{36}$$

where $\hat{d}_i$ is the ith eigenvalue of $\widehat{\mathbf{\Sigma}}$. Expression (36) indicates that shrinking the covariance matrix toward the identity will increase the eigenvalues of the sample covariance matrix, provided that $\bar{\sigma}^2 - \hat{d}_i > 0$, and this effect will be larger among the lowest eigenvalues. Therefore, by shrinking the sample covariance matrix we attenuate the impact that low-variance principal components have on the out-of-sample performance of sample portfolios.

In Figure 5, we exploit the shrinkage covariance matrix (34) to illustrate the impact that low-variance principal components have on the OOSU risk and robustness of the SMV portfolio. For each dataset, we draw 10,000 bootstrap samples of $2T$ observations, where $T = 120$. We estimate the SMV portfolio exploiting the shrinkage covariance matrix $\widehat{\mathbf{\Sigma}}_{\text{sh}}$ using the first T observations. Then, we evaluate the OOSU of this portfolio with $\gamma = 3$ in the second half of the sample. The left axis plots OOSU risk, which is the standard deviation

Figure 5: Shrinkage covariance matrix and out-of-sample utility of SMV portfolio

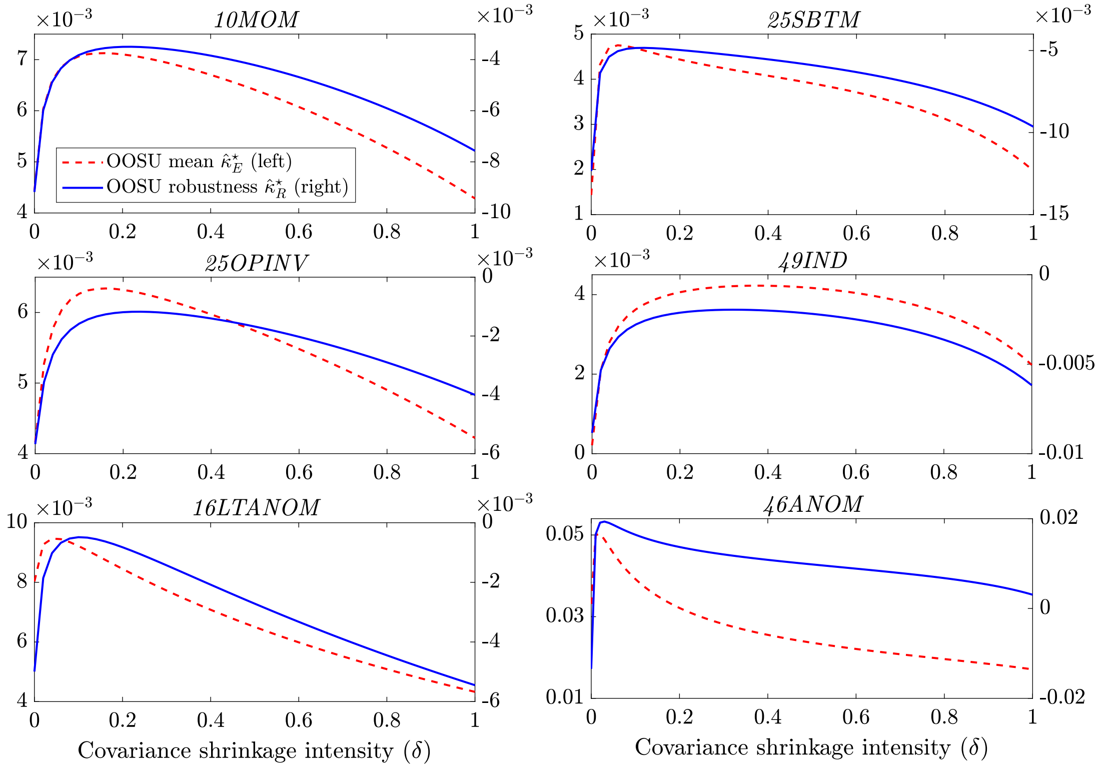


Notes. This figure depicts the out-of-sample utility standard deviation (left axis) and the portfolio robustness measure in Section 6 (right axis) for the sample mean-variance (SMV) portfolio as a function of the intensity with which the sample covariance matrix is shrunk toward the identity. More precisely, we use the covariance matrix $\widehat{\mathbf{\Sigma}}_{\text{sh}} = (1-\delta)\widehat{\mathbf{\Sigma}} + \delta\bar{\sigma}^2\mathbf{I}$ where $\bar{\sigma}^2$ is the cross-sectional average of asset-return variances and δ is the shrinkage intensity. For each dataset, we draw 10,000 bootstrap samples of $2T$ observations (with replacement), where $T = 120$. We estimate the SMV portfolio using the first T observations and the shrinkage covariance matrix $\widehat{\mathbf{\Sigma}}_{\text{sh}}$. Then, we evaluate the OOSU of this portfolio in the second half of the sample. Finally, we compute the OOSU risk and robustness of the SMV portfolio across the 10,000 bootstrap samples. We run this experiment for a shrinkage intensity δ between zero and one. We consider a risk-aversion coefficient of $\gamma = 3$ and a coefficient $\lambda = 2$ for the portfolio robustness measure.

of the OOSU of the SMV portfolio across all 10,000 bootstrap samples. Similarly, the right axis plots the mean-risk OOSU with $\lambda = 2$. We run this experiment for different values of the shrinkage intensity $\delta \in [0, 1]$.

Even though low-variance principal components help improve *in-sample* performance (Kozak et al., 2018), the empirical exercise in Figure 5 corroborates the results in Propositions 4 and 5 that exploiting low-variance principal components contributes to having larger OOSU risk. Moreover, Figure 5 also shows that shrinking the sample covariance matrix helps decrease OOSU risk and increase the robustness of SMV portfolios. In addition to the previous analysis, we are also interested in whether shrinking the sample covariance matrix can

Figure 6: Shrinkage covariance matrix and out-of-sample utility of shrinkage portfolios



Notes. This figure depicts the out-of-sample utility mean of the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ (left axis) and the portfolio robustness measure in Section 6 of the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ (right axis) as a function of the intensity with which the sample covariance matrix is shrunk toward the identity. More precisely, we use the covariance matrix $\widehat{\mathbf{\Sigma}}_{\text{sh}} = (1-\delta)\widehat{\mathbf{\Sigma}} + \delta\bar{\sigma}^2\mathbf{I}$ where $\bar{\sigma}^2$ is the cross-sectional average of asset-return variances and δ is the shrinkage intensity. For each dataset, we draw 10,000 bootstrap samples of $2T$ observations (with replacement), where $T = 120$. We estimate the two shrinkage portfolios using the first T observations and the shrinkage covariance matrix $\widehat{\mathbf{\Sigma}}_{\text{sh}}$. Then, we evaluate the OOSU of the two portfolios in the second half of the sample. Finally, we compute the OOSU mean and robustness across the 10,000 bootstrap samples. We run this experiment for a shrinkage intensity δ between zero and one. We consider a risk-aversion coefficient of $\gamma = 3$ and a coefficient $\lambda = 2$ for the portfolio robustness measure.

further improve the performance of shrinkage portfolios that optimally combine the SMV and SGMV portfolios.

Figure 6 illustrates this by depicting the relationship between the shrinkage intensity of the covariance matrix and the out-of-sample performance of the shrinkage portfolios that optimally combine the SMV and SGMV portfolios. For the shrinkage portfolio that maximizes OOSU mean (i.e., using intensity $\hat{\kappa}_E^\star$), we depict the relation between the shrinkage intensity of the covariance matrix and the OOSU mean. For the shrinkage portfolio that maximizes the mean-risk OOSU with $\lambda = 2$ (i.e., using intensity $\hat{\kappa}_R^\star$), we depict the relation between the shrinkage intensity of the covariance matrix and the mean-risk OOSU. The figure shows across all datasets that by shrinking the sample covariance matrix, we can further improve the out-of-sample performance of shrinkage portfolios. In the next section, we capitalize on this insight and construct the shrinkage portfolios using a shrinkage covariance matrix as in (34) where the shrinkage intensity δ is calibrated using the same criterion used to obtain the optimal combination between the SMV and SGMV portfolios.

7.3 Empirical returns

In this section, we evaluate the out-of-sample performance of the robust portfolio and several benchmark strategies using empirical return data. Consistent with the analysis with simulated data, the results in this section confirm that our proposed shrinkage portfolio is a robust strategy that delivers favorable average out-of-sample performance *and* a stable out-of-sample performance.

We use the six datasets of characteristic and industry-sorted portfolios described in Section 7.1. We study the performance of six portfolio strategies. First, the equally weighted (EW) portfolio. Second, the reward-to-risk (RTR) timing strategy of Kirby and Ostdiek (2012). Third, the sample global-minimum-variance (SGMV) portfolio. Fourth, the sample mean-variance (SMV) portfolio. Fifth, the shrinkage portfolio that exploits the intensity $\hat{\kappa}_E^\star$ maximizing OOSU mean as in Kan et al. (2021b). The last portfolio strategy is our proposed shrinkage portfolio that exploits the intensity $\hat{\kappa}_R^\star$ maximizing the mean-risk OOSU criterion that we introduce in Section 6, with a fixed value of $\lambda = 2$ as in the simulated return data experiment of Section 7.2. We set the risk-aversion coefficient to $\gamma = 3$ as in Kan and Zhou (2007) and Kan et al. (2021b).[17]

Our performance analysis exploits a covariance matrix that shrinks the sample covariance matrix toward the identity. In particular, the SMV and SGMV portfolios are constructed using the shrinkage covariance matrix of Ledoit and Wolf (2004), and the shrinkage portfolios use a covariance matrix whose shrinkage intensity is calibrated to optimize the same criterion used to combine the SGMV and SMV portfolios. In particular, for the shrinkage portfolio that maximizes OOSU mean, we numerically find the shrinkage intensity of the

[17]In Appendix IA.4.3, we also consider risk aversions of $\gamma = 1$ and 5. In addition, in Appendix IA.4.5, we consider a shrinkage portfolio that combines the SMV, SGMV, and EW portfolios as in Tu and Zhou (2011).

covariance matrix that maximizes the portfolio's OOSU mean. For the shrinkage portfolio that maximizes the mean-risk OOSU with $\lambda = 2$, we numerically find the shrinkage intensity of the covariance matrix that maximizes the portfolio's mean-risk OOSU.[18] While this empirical setting imposes a higher hurdle for our proposed robust portfolio, we still find that our methodology consistently outperforms.[19]

Similar to DeMiguel et al. (2009b), we use a rolling-window approach to evaluate the out-of-sample performance of the different portfolio strategies. In particular, let τ be the total number of monthly returns in the dataset and T the sample size used to estimate the portfolios. Then, starting in month $T+1$, we estimate portfolio w using an estimation window containing the first T monthly returns of our sample, and compute its out-of-sample return in month $T+1$ as $\tilde{p}_{T+1} = w^\top r_{T+1}$, where r_{T+1} is the vector of stock returns in month $T+1$. We then move the estimation window one month ahead and construct the out-of-sample return in month $T+2$ similarly. We repeat this process until the end of the sample, which gives a time series of $\tau - T$ out-of-sample returns, i.e., $\tilde{p}_t$, $t = T+1, \ldots, \tau$. Our experiments consider estimation window sizes of $T = 120$ and 240 monthly observations.

We compute the portfolio turnover at time t as

$$\text{Turnover}_t = \sum_{i=1}^{N} |w_{i,t} - w_{i,(t-1)^+}|, \quad t = T+1, \ldots, \tau, \tag{37}$$

where $w_{i,t}$ is the weight of stock i in month t and $w_{i,(t-1)^+}$ is the prior-month weight before rebalancing in month t that takes into account portfolio growth. We use the portfolio turnover at time t to compute out-of-sample portfolio returns net of proportional transaction costs as

$$p_{T+1} = \tilde{p}_{T+1} \quad \text{and} \quad p_t = (1+\tilde{p}_t)(1 - c \times \text{Turnover}_{t-1}) - 1, \quad t = T+2, \ldots, \tau, \tag{38}$$

[18]Provided a shrinkage intensity δ for the shrinkage covariance matrix used in the construction of the SMV and SGMV portfolios, and a shrinkage intensity κ for the optimal combination of the SMV and SGMV portfolios, we draw 1,000 bootstrap samples of T observations (with replacement) from the estimation window. For each bootstrap sample, we approximate the OOSU of the shrinkage portfolios exploiting the shrinkage covariance matrix as the certainty-equivalent return of the out-of-sample returns from a five-fold cross-validation experiment. Finally, we compute the OOSU mean and mean-risk OOSU of the estimated shrinkage portfolios across the 1,000 bootstrap samples using (31) and (33). We find numerically the value of the shrinkage intensity δ for the shrinkage covariance matrix that maximizes the OOSU mean of the shrinkage portfolio exploiting $\hat{\kappa}^\star_E$ and that maximizes the mean-risk OOSU of the shrinkage portfolio exploiting $\hat{\kappa}^\star_R$.

[19]In unreported results, we find that our empirical findings are robust to using the sample covariance matrix or the non-linear shrinkage covariance matrix of Ledoit and Wolf (2017).

where c is the proportional cost required to rebalance the portfolio. We report the results for the case without transaction costs (i.e., $c = 0$), and for the case where $c = 20$ basis points, which is the same level of transaction costs as in Kan et al. (2021b).[20]

We then compute the out-of-sample mean and variance of portfolio returns net of costs as

$$\mu_p = \frac{1}{\tau - T} \sum_{t=T+1}^{\tau} p_t \quad \text{and} \quad \sigma_p^2 = \frac{1}{\tau - T} \sum_{t=T+1}^{\tau} (p_t - \mu_p)^2.$$

We compare the six portfolio strategies in terms of their annualized out-of-sample certainty-equivalent return (CER) and Sharpe ratio (SR):

$$\text{OOS CER} = 12 \times \left(\mu_p - \frac{\gamma}{2}\sigma_p^2\right), \tag{39}$$

$$\text{OOS SR} = \sqrt{12} \times \mu_p / \sigma_p. \tag{40}$$

The OOS CER corresponds to the empirical out-of-sample utility of estimated portfolios. We also test the null hypothesis that the OOS CER or SR delivered by $\hat{\kappa}_R^\star$ are equal to those delivered by $\hat{\kappa}_E^\star$ and the EW portfolio, against the alternative hypothesis that $\hat{\kappa}_R^\star$ yields larger OOS CER or SR. To compute the test p-values, we generate 1,000 bootstrap samples using the stationary block bootstrap approach of Politis and Romano (1994) with an average block size of five and use the methodology of Ledoit and Wolf (2008, Remark 3.2.) to produce the resulting p-values.

Finally, we also evaluate the out-of-sample performance risk of shrinkage portfolios by dividing the out-of-sample portfolio returns into non-overlapping three-year windows.[21] We then compute the OOS CER in each window, and the mean and standard deviation of the OOS CER across all windows. This allows us to evaluate the average out-of-sample performance delivered by the shrinkage portfolios and their out-of-sample performance risk, in line with the robustness measure we introduce in this manuscript.

Table 2 reports the out-of-sample results for the portfolios constructed with the standard estimation window of $T = 120$ monthly observations. We document that the proposed robust

[20] In Appendix IA.4.4, we show that our conclusions are robust to considering a larger $c = 30$ basis points.

[21] We use three-year windows to obtain a reasonable tradeoff between performance evaluation frequency and the number of observations in each window. The conclusions are robust to using other window lengths.

shrinkage portfolio outperforms in terms of certainty-equivalent return, which corresponds to the empirical out-of-sample utility of estimated portfolios, and Sharpe ratio. In particular, the median improvement in terms of certainty-equivalent return (Sharpe ratio) net of transaction costs across the six datasets is 79% (29%) relative to the shrinkage portfolio only maximizing OOSU mean, 151% (30%) relative to the SMV portfolio, 50% (21%) relative to the SGMV portfolio, 100% (51%) relative to the timing portfolio, and 179% (74%) relative to the equally weighted portfolio. The outperformance of the robust shrinkage portfolio relative to the benchmark portfolios is similar in magnitude in the absence of transaction costs.

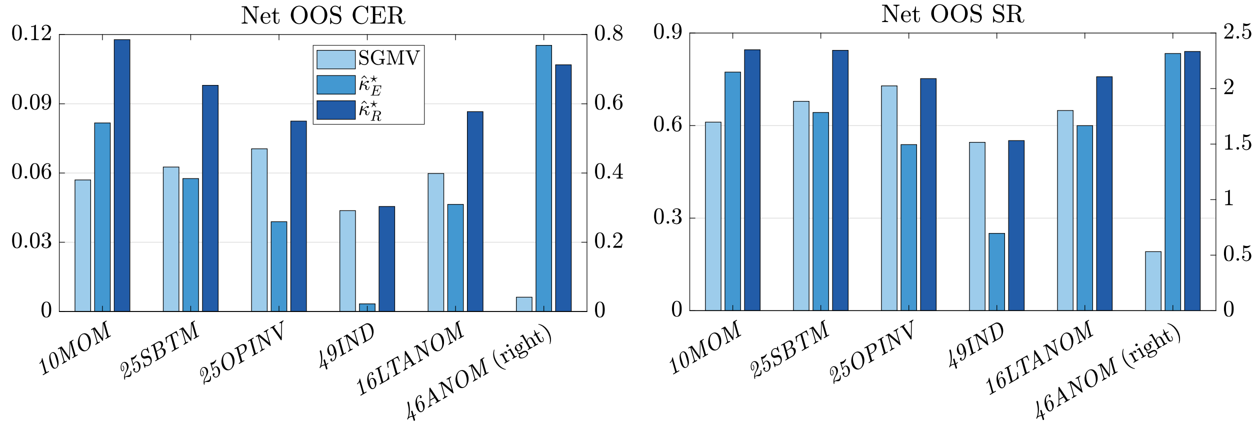
Table 3 reports the out-of-sample results for the portfolios constructed with a large estimation window of $T = 240$ monthly observations. The results in this new experimental setting are consistent to those presented in Table 2. In particular, the median improvement in terms of certainty-equivalent return (Sharpe ratio) net of transaction costs across the six datasets is 66% (16%) relative to the shrinkage portfolio only maximizing OOSU mean, 75% (17%) relative to the SMV portfolio, 62% (13%) relative to the SGMV portfolio, 124% (50%) relative to the timing portfolio and 203% (80%) relative to the equally weighted portfolio. Like in the case with $T = 120$, the robust shrinkage portfolio also outperforms the benchmark portfolios in the absence of transaction costs for $T = 240$.

Figure 7 presents an overview of the results for the case with an estimation window of $T = 120$ monthly observations. In particular, this figure depicts the certainty-equivalent return (OOS CER) and Sharpe ratio (OOS SR) net of transaction costs. From this figure, it is easy to appreciate the favorable performance exhibited by our robust shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ relative to the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ and the SGMV portfolio.

Finally, in addition to outperforming the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ in terms of average OOS CER, the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ also delivers a substantially more stable out-of-sample performance. To see this, we report in Table 4 the mean and standard deviation of out-of-sample CER across three-year non-overlapping windows. We find that the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ systematically delivers an OOS CER that is both larger on average and more stable over time. This result is consistent with our definition of portfolio robustness in the presence of parameter uncertainty.

In Appendix IA.4, we confirm that our insights are robust to considering the tail risk

Figure 7: Out-of-sample certainty-equivalent return and Sharpe ratio in empirical data



Notes. This figure depicts the annualized out-of-sample certainty-equivalent return and Sharpe ratio of the SGMV portfolio, the shrinkage portfolio maximizing out-of-sample utility mean ($\hat{\kappa}_E^\star$), and the shrinkage portfolio maximizing the portfolio robustness measure in Section 6 ($\hat{\kappa}_R^\star$) for the six datasets of monthly excess returns described in Section 7.1. We use a sample size of $T = 120$ monthly observations and a risk-aversion coefficient of $\gamma = 3$. The certainty-equivalent return and Sharpe ratio are net of proportional transaction costs of 20 basis points. The right vertical axis reports the performance for the *46ANOM* dataset for visibility.

of the portfolios, the cumulative wealth derived from the portfolios, different risk-aversion coefficients, a higher level of transaction costs, and a different combination of portfolios that exploits the SMV, SGMV, and equally weighted portfolios as in Tu and Zhou (2011).

8 Conclusion

In this manuscript, we characterize the out-of-sample utility (OOSU) risk of the sample mean-variance (SMV) portfolio, the sample global-minimum-variance (SGMV) portfolio and any linear combination of the two portfolios. We show that SMV portfolios need unrealistically large sample sizes —in some cases over 1,000 years of monthly return data— to deliver an out-of-sample performance as stable as that of SGMV portfolios. In general, the OOSU risk of sample portfolios is substantial in high-dimensional settings where the number of assets is large relative to the sample size, and when these portfolios exploit in-sample near-arbitrage opportunities. We then use our characterization of OOSU risk to propose a novel measure of portfolio robustness that strikes a balance between OOSU mean and OOSU volatility. We show that neither the SMV portfolio nor the SGMV portfolio delivers maximal robustness individually. In particular, one needs to *optimally combine* both portfolios to obtain an improved tradeoff between OOSU mean and volatility. We find that shrinkage portfolios that optimize our proposed measure of portfolio robustness deliver higher certainty-equivalent returns and Sharpe ratios.

Our robust portfolio framework can be applied to a broader range of settings than the one considered in the main body of the manuscript. For instance, we also use our framework to construct shrinkage portfolios that combine the SMV portfolio, the SGMV portfolio, and the equally weighted portfolio as in Tu and Zhou (2011). Regardless of the combination used in the empirical analysis, we show that our methodology offers robust shrinkage portfolios that are resilient to estimation errors and exhibit favorable performance.

Our analysis of OOSU risk has implications for cross-sectional asset pricing. In particular, we link the OOSU of estimated portfolios with the out-of-sample R-squared of SDF models. Our theoretical results suggest that SDF models constructed from a large number of test assets with short time series and that capture near-arbitrage opportunities will deliver, in general, unreliable results due to their large out-of-sample R-squared risk. In contrast, SDF models built from our robust approach will deliver more reliable results with lower out-of-sample R-squared risk. More broadly, our analysis highlights the importance of considering the out-of-sample performance risk in the evaluation and construction of quantitative strategies and models of the stochastic discount factor.

Tables

Table 1: Out-of-sample performance of shrinkage portfolios in simulated data

		OOSU mean			OOSU risk			Mean-risk OOSU		
	T	120	180	240	120	180	240	120	180	240
Panel A: Gaussian data (in %)										
10MOM	$\hat{\kappa}^\star_E$	0.63	0.72	0.77	0.27	0.18	0.15	0.09	0.35	0.47
	$\hat{\kappa}^\star_R$	0.66	0.72	0.76	0.20	0.15	0.13	0.27	0.43	0.50
25SBTM	$\hat{\kappa}^\star_E$	0.45	0.61	0.71	0.37	0.24	0.20	-0.28	0.13	0.31
	$\hat{\kappa}^\star_R$	0.50	0.62	0.70	0.25	0.19	0.17	0.01	0.24	0.36
25OPINV	$\hat{\kappa}^\star_E$	0.67	0.85	0.98	0.38	0.27	0.23	-0.10	0.32	0.52
	$\hat{\kappa}^\star_R$	0.70	0.84	0.95	0.26	0.22	0.21	0.17	0.40	0.52
49IND	$\hat{\kappa}^\star_E$	0.33	0.56	0.70	0.44	0.29	0.23	-0.56	-0.02	0.24
	$\hat{\kappa}^\star_R$	0.40	0.57	0.69	0.26	0.20	0.19	-0.11	0.17	0.31
16LTANOM	$\hat{\kappa}^\star_E$	1.11	1.37	1.54	0.42	0.34	0.29	0.27	0.70	0.97
	$\hat{\kappa}^\star_R$	1.06	1.32	1.50	0.37	0.34	0.31	0.32	0.63	0.89
46ANOM	$\hat{\kappa}^\star_E$	5.84	8.04	9.31	1.48	1.11	0.91	2.89	5.81	7.48
	$\hat{\kappa}^\star_R$	5.70	7.96	9.27	1.27	1.03	0.87	3.17	5.90	7.54
Panel B: Boostrapped data (in %)										
10MOM	$\hat{\kappa}^\star_E$	0.45	0.54	0.61	0.93	0.63	0.49	-1.41	-0.72	-0.36
	$\hat{\kappa}^\star_R$	0.53	0.58	0.63	0.77	0.54	0.43	-1.01	-0.50	-0.22
25SBTM	$\hat{\kappa}^\star_E$	0.14	0.29	0.43	1.05	0.72	0.57	-1.96	-1.16	-0.71
	$\hat{\kappa}^\star_R$	0.31	0.38	0.48	0.77	0.57	0.46	-1.23	-0.76	-0.45
25OPINV	$\hat{\kappa}^\star_E$	0.37	0.50	0.59	0.82	0.56	0.41	-1.27	-0.61	-0.24
	$\hat{\kappa}^\star_R$	0.47	0.55	0.61	0.63	0.44	0.34	-0.80	-0.33	-0.06
49IND	$\hat{\kappa}^\star_E$	0.07	0.18	0.23	0.69	0.44	0.35	-1.31	-0.69	-0.47
	$\hat{\kappa}^\star_R$	0.21	0.27	0.30	0.50	0.33	0.27	-0.78	-0.39	-0.24
16LTANOM	$\hat{\kappa}^\star_E$	0.75	0.99	1.16	0.85	0.62	0.48	-0.95	-0.24	0.21
	$\hat{\kappa}^\star_R$	0.75	0.96	1.12	0.67	0.52	0.42	-0.60	-0.08	0.29
46ANOM	$\hat{\kappa}^\star_E$	3.53	4.95	5.93	3.59	2.71	2.05	-3.65	-0.46	1.84
	$\hat{\kappa}^\star_R$	3.90	5.24	6.13	2.61	2.19	1.71	-1.32	0.85	2.70

Notes. This table reports the out-of-sample performance of estimated shrinkage portfolios across six different datasets of simulated data. The first two blocks of three columns report the mean and standard deviation of the monthly out-of-sample utility (in percentage) of the estimated shrinkage portfolios. The third block of three columns reports the mean-risk out-of-sample utility, which is the proposed robustness metric in Section 6. We report the results for the estimated shrinkage portfolio maximizing out-of-sample utility mean ($\hat{\kappa}^\star_E$) and for the estimated shrinkage portfolio maximizing the proposed robustness measure ($\hat{\kappa}^\star_R$). In Panel A, we define the population parameters of a multivariate Gaussian distribution with the sample moments of each of the six datasets of monthly excess returns described in Section 7.1, and draw 100,000 samples of size T. For each simulated sample, we construct the two shrinkage portfolios and their corresponding out-of-sample utilities using Equation (12). Finally, we use the out-of-sample utilities of the 100,000 simulated samples to construct our performance metrics as in (31)-(33). We proceed similarly in Panel B by generating 1,000 bootstrap samples of size $2T$ monthly observations, constructing the two shrinkage portfolios using the first T observations and computing the out-of-sample utility in the remaining T observations, and finally evaluating the performance metrics across the 1,000 simulated bootstrap samples. We consider sample sizes of $T = 120$, 180 and 240 monthly observations, a risk-aversion coefficient of $\gamma =\ 3$, and a coefficient $\lambda = 2$ for the portfolio robustness measure.

Table 2: Out-of-sample performance in empirical data ($T = 120$)

		EW	RTR	SGMV	SMV	$\hat{\kappa}_E^\star$	$\hat{\kappa}_R^\star$
10MOM	Gross OOS CER	0.034	0.051	0.061	0.082	0.115	$\mathbf{0.134}_{\circ\circ\circ}$
	Net OOS CER	0.033	0.049	0.057	0.047	0.082	$\mathbf{0.118}^{\star\star}_{\circ\circ\circ}$
	Gross OOS SR	0.455	0.558	0.641	0.877	0.865	$\mathbf{0.909}_{\circ\circ\circ}$
	Net OOS SR	0.449	0.547	0.611	0.798	0.773	$\mathbf{0.846}^{\star\star}_{\circ\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.395	0.296
25SBTM	Gross OOS CER	0.045	0.052	0.072	-0.012	**0.123**	$\mathbf{0.123}_{\circ\circ\circ}$
	Net OOS CER	0.044	0.051	0.063	-0.107	0.058	$\mathbf{0.098}^{\star\star}_{\circ\circ\circ}$
	Gross OOS SR	0.522	0.561	0.748	0.813	0.861	$\mathbf{0.991}^{\star\star}_{\circ\circ\circ}$
	Net OOS SR	0.516	0.554	0.679	0.631	0.643	$\mathbf{0.844}^{\star\star\star}_{\circ\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.214	0.145
25OPINV	Gross OOS CER	0.044	0.055	0.079	-0.269	0.080	$\mathbf{0.102}_{\circ\circ\circ}$
	Net OOS CER	0.043	0.053	0.071	-0.364	0.039	$\mathbf{0.083}^{\star\star\star}_{\circ\circ}$
	Gross OOS SR	0.515	0.586	0.794	0.614	0.699	$\mathbf{0.872}^{\star\star\star}_{\circ\circ\circ}$
	Net OOS SR	0.509	0.575	0.729	0.461	0.538	$\mathbf{0.752}^{\star\star\star}_{\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.191	0.120
49IND	Gross OOS CER	0.050	0.052	0.053	-1.456	0.039	$\mathbf{0.057}^{\star\star}$
	Net OOS CER	0.049	**0.050**	0.044	-1.565	0.008	$0.046^{\star\star\star}$
	Gross OOS SR	0.552	0.565	0.624	0.264	0.486	$\mathbf{0.645}^{\star\star\star}$
	Net OOS SR	0.544	**0.554**	0.546	0.125	0.279	$0.551^{\star\star\star}$
	Average $\hat{\kappa}$	/	/	0	1	0.043	0.024
16LTANOM	Gross OOS CER	0.027	0.044	0.066	0.016	0.098	$\mathbf{0.109}_{\circ\circ\circ}$
	Net OOS CER	0.026	0.042	0.060	-0.040	0.046	$\mathbf{0.087}^{\star\star}_{\circ\circ}$
	Gross OOS SR	0.419	0.516	0.695	0.734	0.775	$\mathbf{0.887}^{\star}_{\circ\circ\circ}$
	Net OOS SR	0.413	0.505	0.649	0.609	0.600	$\mathbf{0.758}^{\star\star}_{\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.309	0.218
46ANOM	Gross OOS CER	0.016	0.052	0.054	-0.532	**0.831**	$0.765_{\circ\circ\circ}$
	Net OOS CER	0.015	0.051	0.041	-0.609	**0.769**	$0.713_{\circ\circ\circ}$
	Gross OOS SR	0.358	0.566	0.643	2.272	2.431	$\mathbf{2.442}_{\circ\circ\circ}$
	Net OOS SR	0.353	0.555	0.532	2.134	2.316	$\mathbf{2.334}_{\circ\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.250	0.207

Notes. This table reports the out-of-sample performance of the six portfolio strategies described in Section 7.3 for the six datasets of monthly excess returns described in Section 7.1. Each estimated portfolio is constructed using a sample size of $T = 120$ monthly observations. The mean-variance portfolios consider a risk-aversion coefficient of $\gamma = 3$. The table reports the annualized out-of-sample certainty-equivalent return (OOS CER) and the annualized out-of-sample Sharpe ratio (OOS SR), and for both criteria we report the gross performance and the performance net of proportional transaction costs of 20 basis points. We also report the average estimated shrinkage intensity $\hat{\kappa}$ over time, except for the EW and RTR portfolios that do not combine the SMV and SGMV portfolios. The stars $\star, \star\star, \star\star\star$ establish that the OOS CER and SR of the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ is larger than that of the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ at a confidence level of 10%, 5%, and 1%, respectively. The circles $\circ, \circ\circ, \circ\circ\circ$ convey the same information relative to the EW portfolio. The numbers in bold font identify the best portfolio in terms of OOS CER and SR.

Table 3: Out-of-sample performance in empirical data ($T = 240$)

		EW	RTR	SGMV	SMV	$\hat{\kappa}_E^\star$	$\hat{\kappa}_R^\star$
10MOM	Gross OOS CER	0.039	0.057	0.069	0.106	0.134	$\mathbf{0.152}_{\circ\circ\circ}$
	Net OOS CER	0.038	0.056	0.065	0.083	0.114	$\mathbf{0.138}^{\star\star}_{\circ\circ\circ}$
	Gross OOS SR	0.484	0.607	0.706	0.917	0.917	$\mathbf{0.959}^{\star}_{\circ\circ\circ}$
	Net OOS SR	0.478	0.600	0.682	0.864	0.862	$\mathbf{0.912}^{\star\star}_{\circ\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.546	0.471
25SBTM	Gross OOS CER	0.053	0.060	0.081	0.092	0.139	$\mathbf{0.144}_{\circ\circ\circ}$
	Net OOS CER	0.052	0.059	0.074	0.031	0.095	$\mathbf{0.111}_{\circ\circ\circ}$
	Gross OOS SR	0.567	0.612	0.833	0.917	0.914	$\mathbf{1.012}^{\star\star}_{\circ\circ\circ}$
	Net OOS SR	0.561	0.606	0.778	0.788	0.776	$\mathbf{0.851}^{\star}_{\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.364	0.289
25OPINV	Gross OOS CER	0.050	0.060	0.102	-0.103	0.124	$\mathbf{0.135}_{\circ\circ\circ}$
	Net OOS CER	0.049	0.059	0.097	-0.154	0.099	$\mathbf{0.119}^{\star}_{\circ\circ}$
	Gross OOS SR	0.556	0.633	0.976	0.660	0.872	$\mathbf{1.017}^{\star\star\star}_{\circ\circ\circ}$
	Net OOS SR	0.550	0.627	**0.935**	0.565	0.772	$0.928^{\star\star\star}_{\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.320	0.232
49IND	Gross OOS CER	0.052	**0.054**	0.044	-0.784	0.024	$0.046^{\star\star}$
	Net OOS CER	0.051	**0.053**	0.038	-0.846	0.008	$0.036^{\star\star\star}$
	Gross OOS SR	0.569	**0.592**	0.557	0.111	0.391	$0.557^{\star\star}$
	Net OOS SR	0.562	**0.585**	0.501	0.014	0.293	$0.475^{\star\star}$
	Average $\hat{\kappa}$	/	/	0	1	0.087	0.054
16LTANOM	Gross OOS CER	0.028	0.045	0.095	0.123	0.172	$\mathbf{0.175}_{\circ\circ\circ}$
	Net OOS CER	0.027	0.044	0.091	0.086	0.136	$\mathbf{0.158}_{\circ\circ\circ}$
	Gross OOS SR	0.422	0.520	0.942	0.947	1.016	$\mathbf{1.132}^{\star}_{\circ\circ\circ}$
	Net OOS SR	0.416	0.513	0.904	0.863	0.910	$\mathbf{1.051}^{\star\star}_{\circ\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.522	0.449
46ANOM	Gross OOS CER	0.022	0.057	0.098	-2.160	0.138	$\mathbf{0.347}^{\star\star\star}_{\circ\circ}$
	Net OOS CER	0.021	0.056	0.088	-2.159	0.022	$\mathbf{0.295}^{\star\star\star}_{\circ}$
	Gross OOS SR	0.399	0.600	1.037	1.522	1.536	$\mathbf{1.648}^{\star\star}_{\circ\circ\circ}$
	Net OOS SR	0.393	0.593	0.945	1.427	1.371	$\mathbf{1.573}^{\star\star\star}_{\circ\circ\circ}$
	Average $\hat{\kappa}$	/	/	0	1	0.511	0.471

Notes. This table reports the out-of-sample performance of the six portfolio strategies described in Section 7.3 for the six datasets of monthly excess returns described in Section 7.1. Each estimated portfolio is constructed using a sample size of $T = 240$ monthly observations. The mean-variance portfolios consider a risk-aversion coefficient of $\gamma = 3$. The table reports the annualized out-of-sample certainty-equivalent return (OOS CER) and the annualized out-of-sample Sharpe ratio (OOS SR), and for both criteria we report the gross performance and the performance net of proportional transaction costs of 20 basis points. We also report the average estimated shrinkage intensity $\hat{\kappa}$ over time, except for the EW and RTR portfolios that do not combine the SMV and SGMV portfolios. The stars $\star, \star\star, \star\star\star$ establish that the OOS CER and SR of the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ is larger than that of the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ at a confidence level of 10%, 5%, and 1%, respectively. The circles $\circ, \circ\circ, \circ\circ\circ$ convey the same information relative to the EW portfolio. The numbers in bold font identify the best portfolio in terms of OOS CER and SR.

Table 4: Out-of-sample certainty-equivalent returns of shrinkage portfolios in empirical data

		Mean OOS CER		Std dev OOS CER	
		$T = 120$	$T = 240$	$T = 120$	$T = 240$
Panel A: Without transaction costs (in %)					
10MOM	$\hat{\kappa}^{\star}_{E}$	1.027	1.205	2.444	2.631
	$\hat{\kappa}^{\star}_{R}$	1.138	1.312	1.576	2.031
25SBTM	$\hat{\kappa}^{\star}_{E}$	1.074	1.230	1.555	1.739
	$\hat{\kappa}^{\star}_{R}$	1.028	1.255	1.071	1.051
25OPINV	$\hat{\kappa}^{\star}_{E}$	0.613	0.945	1.315	1.198
	$\hat{\kappa}^{\star}_{R}$	0.831	1.118	0.997	0.888
49IND	$\hat{\kappa}^{\star}_{E}$	0.361	0.252	1.178	0.927
	$\hat{\kappa}^{\star}_{R}$	0.522	0.425	0.841	0.786
16LTANOM	$\hat{\kappa}^{\star}_{E}$	0.813	1.448	1.497	2.458
	$\hat{\kappa}^{\star}_{R}$	0.925	1.443	1.12	1.722
46ANOM	$\hat{\kappa}^{\star}_{E}$	7.182	2.709	6.423	4.821
	$\hat{\kappa}^{\star}_{R}$	6.509	4.287	5.646	5.534
Panel B: Net of transaction costs (in %)					
10MOM	$\hat{\kappa}^{\star}_{E}$	0.743	1.037	2.422	2.628
	$\hat{\kappa}^{\star}_{R}$	1.002	1.201	1.565	2.040
25SBTM	$\hat{\kappa}^{\star}_{E}$	0.515	0.861	1.362	1.769
	$\hat{\kappa}^{\star}_{R}$	0.819	0.977	1.012	1.002
25OPINV	$\hat{\kappa}^{\star}_{E}$	0.260	0.737	1.339	1.188
	$\hat{\kappa}^{\star}_{R}$	0.666	0.986	1.001	0.878
49IND	$\hat{\kappa}^{\star}_{E}$	0.098	0.113	1.170	0.943
	$\hat{\kappa}^{\star}_{R}$	0.423	0.344	0.834	0.807
16LTANOM	$\hat{\kappa}^{\star}_{E}$	0.390	1.162	1.478	2.438
	$\hat{\kappa}^{\star}_{R}$	0.740	1.303	1.109	1.730
46ANOM	$\hat{\kappa}^{\star}_{E}$	6.645	1.597	6.207	4.996
	$\hat{\kappa}^{\star}_{R}$	6.064	3.762	5.437	5.364

Notes. This table reports the out-of-sample certainty-equivalent return (OOS CER) mean and standard deviation. We consider the estimated shrinkage portfolio that maximizes out-of-sample utility mean ($\hat{\kappa}^{\star}_{E}$), and the estimated shrinkage portfolio that maximizes the portfolio robustness measure in Section 6 ($\hat{\kappa}^{\star}_{R}$). We report the performance results of the shrinkage portfolios for the six datasets of monthly excess returns described in Section 7.1. We estimate the shrinkage portfolios with sample sizes of $T = 120$ and $T = 240$ monthly observations, a risk-aversion coefficient of $\gamma = 3$, and a coefficient $\lambda = 2$ for the shrinkage portfolio that maximizes the proposed robustness measure. We obtain the performance measures applying Equations (31)-(33) to the OOS CER's of the shrinkage portfolios obtained by dividing the out-of-sample portfolio returns into non-overlapping three-year windows and computing for each three-year window the OOS CER of the shrinkage portfolios. Panel A considers the case without transaction costs, and Panel B considers the case with proportional transaction costs of 20 basis points.

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Internet Appendix to

The Risk of Out-of-Sample Portfolio Performance

In this Internet Appendix, we provide several extensions of our analysis and the proofs for our theoretical results. In Section IA.1, we show the theoretical relationship between the shrinkage portfolio approach considered in the main body of the manuscript and ambiguity-averse portfolios. In Section IA.2, we provide the link between our theory and cross-sectional asset pricing. In Section IA.3, we give details for the feasible estimators we use in the empirical analysis. In Section IA.4, we check the robustness of our main results to several variations of the experimental setting considered in the main body of the manuscript. In Section IA.5, we report the proofs for all the theoretical results in the manuscript and this appendix.

IA.1 Relation with ambiguity-averse portfolios

In this section, we argue that a shrinkage portfolio that combines the SMV and SGMV portfolios is not only a useful technique to mitigate the impact of estimation risk, but it is also an economically sound approach. In particular, we show that there is an explicit relationship between the shrinkage portfolio considered in this manuscript and the optimal portfolio of an ambiguity-averse investor. The insights provided in this section build on the work of Garlappi et al. (2007), who account for ambiguity by considering a joint uncertainty set for the vector of means. This uncertainty set serves as a constraint in the portfolio problem of a mean-variance investor who solves the following mathematical program

$$\max_{w:w^\top e=1} \min_{\mu} \; w^\top \mu - \frac{\gamma}{2} w^\top \widehat{\mathbf{\Sigma}} w \quad \text{subject to } (\hat{\mu}-\mu)^\top \widehat{\mathbf{\Sigma}}^{-1}(\hat{\mu}-\mu) \leq \varepsilon^2, \tag{IA1}$$

where $(\hat{\mu}-\mu)^\top \widehat{\mathbf{\Sigma}}^{-1}(\hat{\mu}-\mu)$ measures the distance between the sample vector of means $\hat{\mu}$ and the vector of means μ used by the investor to optimize her objective function. Intuitively, a larger degree of ambiguity aversion is equivalent to having a larger value of ε in the constraint of portfolio problem (IA1). Garlappi et al. (2007) show that the closed-form solution of this problem is

$$\hat{w}^\star(\varepsilon) = \frac{1}{\gamma} \widehat{\mathbf{\Sigma}}^{-1} \left(\frac{1}{1+\varepsilon/(\gamma\sigma_P^\star)} \right) \left(\hat{\mu} - \frac{B - \gamma(1+\varepsilon/(\gamma\sigma_P^\star))}{A} e \right), \tag{IA2}$$

where $A = e^\top \widehat{\mathbf{\Sigma}}^{-1} e$, $B = \hat{\mu}^\top \widehat{\mathbf{\Sigma}}^{-1} e$, and parameter $\sigma_P^\star$ is the unique positive real root to a specific fourth-degree polynomial that is monotonically decreasing in ε. Garlappi et al. (2007)

show that the ambiguity-averse portfolio $\hat{w}^\star(\varepsilon)$ converges to the SMV portfolio when $\varepsilon \to 0$, and to the SGMV portfolio when $\varepsilon \to \infty$. Intuitively, for $0 < \varepsilon < \infty$ the ambiguity-averse portfolio $\hat{w}^\star(\varepsilon)$ combines the SMV and the SGMV portfolios, similarly to the shrinkage portfolio $\hat{w}^\star(\kappa)$ in (11). In the next proposition, we characterize the intensity κ of the shrinkage portfolio $\hat{w}^\star(\kappa)$ in (11) as a function of the ambiguity-aversion parameter ε.

Proposition IA.1. *The shrinkage portfolio $\hat{w}^\star(\kappa)$ is equal to the ambiguity-averse portfolio $\hat{w}^\star(\varepsilon)$ in Equation (IA2) when*

$$\kappa = \left(1 + \frac{\varepsilon}{\gamma \sigma_P^\star}\right)^{-1}, \tag{IA3}$$

where the ratio $\varepsilon/\sigma_P^\star$ is monotonically increasing in ε.

Proposition IA.1 provides the explicit link between the shrinkage portfolio considered in the main body of the manuscript and the ambiguity-averse portfolio of Garlappi et al. (2007). In particular, Equation (IA3) shows that a high degree of ambiguity in mean returns (i.e., a higher ε) results in an ambiguity-averse portfolio whose weights lean more strongly toward those of the SGMV portfolio, corresponding to a smaller value of κ. Given an optimally calibrated shrinkage intensity κ, Equation (IA3) allows us to determine the equivalent degree of ambiguity in mean returns that results in an ambiguity-averse portfolio that delivers robust out-of-sample performance.

We show in Proposition 7 that our proposed robustness criterion establishes a larger tilt toward the SGMV portfolio and thus a more considerable degree of ambiguity in mean returns than the traditional shrinkage criterion that only maximizes OOSU mean.[22] In other words, our proposed shrinkage criterion based on the portfolio robustness measure introduced in Section 6 delivers portfolios that are less sensitive to estimation errors in mean returns.

IA.2 Relation with cross-sectional asset pricing

This section shows that our characterization of out-of-sample utility (OOSU) risk can be applied to assess the uncertainty of the out-of-sample fit of a particular robust stochastic

[22]For example, for the 25SBTM dataset considered in Figure 1, we find that the shrinkage intensity $\kappa_E^\star = 0.147$ maximizing OOSU mean corresponds to $\varepsilon = 0.77$, whereas the shrinkage intensity $\kappa_R^\star = 0.0886$ maximizing our proposed robsutness measure for $\lambda = 2$ corresponds to $\varepsilon = 1.36$, which is larger.

discount factor (SDF). In particular, we exploit the well-known link between the SDF and the returns of a mean-variance portfolio (Cochrane, 2005; Kozak et al., 2020) where the estimated mean-variance portfolio $\hat{w}$ corresponds to the SDF loadings. Accordingly, we show that our measure of out-of-sample performance defined in Equation (12) is equivalent to the out-of-sample fit of an SDF model that prices the cross-section of stock returns. We define the OOS R-squared of an SDF model as

$$\hat{R}^2_{OOS} = 1 - \frac{(\mu - \boldsymbol{\Sigma}\hat{w})^\top \boldsymbol{\Sigma}^{-1}(\mu - \boldsymbol{\Sigma}\hat{w})}{\theta^2}, \tag{IA4}$$

where $\theta^2 = \mu^\top \boldsymbol{\Sigma}^{-1}\mu$. Like Kozak et al. (2020), our measure of OOS fit determines how well the SDF model, defined by the SDF loadings $\hat{w}$, explains the vector of out-of-sample mean returns, μ. Now, consider the following scaled OOS R-squared

$$\frac{\theta^2}{2}\hat{R}^2_{OOS} = \hat{w}^\top \mu - \frac{1}{2}\hat{w}^\top \boldsymbol{\Sigma}\hat{w}, \tag{IA5}$$

which corresponds to the OOSU of the estimated portfolio $\hat{w}$ for an investor with risk-aversion coefficient $\gamma = 1$. Like the OOSU, the scaled $\hat{R}^2_{OOS}$ is a random variable because it depends on the estimated SDF loadings $\hat{w}$.

The shrinkage mean-variance portfolio we consider in the main body of the manuscript embeds as a particular case the robust SDF loadings that solve the following constrained quadratic program:

$$\hat{w} = \underset{w:\, w^\top e = 1}{\operatorname{argmin}} \quad (\mu - \widehat{\boldsymbol{\Sigma}}w)^\top \widehat{\boldsymbol{\Sigma}}^{-1}(\mu - \widehat{\boldsymbol{\Sigma}}w) \tag{IA6}$$

$$\text{subject to} \quad \min_{\mu} \quad (\hat{\mu} - \mu)^\top \widehat{\boldsymbol{\Sigma}}^{-1}(\hat{\mu} - \mu) \leq \varepsilon^2. \tag{IA7}$$

The solution to this mathematical program delivers the ambiguity-averse mean-variance portfolio problem in Section IA.1 for an investor with risk-aversion coefficient $\gamma = 1$. Due to the link between constraint (IA7) and the shrinkage intensity κ, which we highlight in Proposition IA.1, the shrinkage portfolios considered in the main body of the manuscript can be interpreted as the robust SDF loadings that minimize the Hansen-Jagannathan distance

(Hansen and Jagannathan, 1997) under the economically motivated constraint faced by ambiguity-averse investors.

Given the link between the economically motivated SDF loadings in problem (IA6) and shrinkage portfolios, we can apply our measure of OOSU risk to evaluate the risk of the OOS fit of robust SDF models. Our theoretical results in Section 5 demonstrate that the OOS R-squared risk is large in high-dimensional settings and when the estimated SDF loadings of the model exploit near-arbitrage opportunities. Accordingly, SDF models constructed from many test assets with short time series and that capture near-arbitrage opportunities are, in general, unreliable.

IA.3 Feasible estimators of shrinkage intensities

The shrinkage intensities $\kappa_E^\star$ and $\kappa_R^\star$ in Equations (17) and (28) are unfeasible because they depend on the unknown distributional parameters of stock returns. In this section, we provide the details of the feasible estimators we use in the performance analysis.

For the return variance of the zero-cost portfolio, which is defined in (7) as ψ^2, we use the adjusted estimator proposed by Kan and Zhou (2007). Let $\hat{\psi}^2 = \hat{\mu}^\top \widehat{\mathbf{B}} \hat{\mu}$ be the plug-in estimator, then we estimate ψ^2 as

$$\hat{\psi}^2_{kz} = \frac{(T-N-1)\hat{\psi}^2 - (N-1)}{T} + \frac{2(\hat{\psi}^2)^{\frac{N-1}{2}}(1+\hat{\psi}^2)^{-\frac{T-2}{2}}}{T \times B\left(\frac{\hat{\psi}^2}{1+\hat{\psi}^2}; \frac{N-1}{2}, \frac{T-N+1}{2}\right)}, \tag{IA8}$$

where $B(x; a, b) = \int_0^x y^{a-1}(1-y)^{b-1}\mathrm{d}y$ is the incomplete beta function.

For the return variance of the GMV portfolio, which is defined as σ_g^2 in (6), we rely on the shrinkage portfolio estimator proposed by Frahm and Memmel (2010, Theorem 2), which gives a smaller mean out-of-sample variance than the SGMV portfolio. Specifically, we estimate σ_g^2 as

$$\hat{\sigma}_g^2 = \hat{w}_{fm}^\top \widehat{\mathbf{\Sigma}} \hat{w}_{fm}, \tag{IA9}$$

where $\hat{w}_{fm}$ combines the equally weighted portfolio and the SGMV portfolio as

$$\hat{w}_{fm} = \hat{\pi}_{fm} w_{ew} + (1 - \hat{\pi}_{fm})\hat{w}_g,$$

with a shrinkage intensity

$$\hat{\pi}_{fm} = \min\left(1, \frac{N-3}{T-N+2}\frac{\hat{w}_g^\top \widehat{\mathbf{\Sigma}} \hat{w}_g}{w_{ew}^\top \widehat{\mathbf{\Sigma}} w_{ew} - \hat{w}_g^\top \widehat{\mathbf{\Sigma}} \hat{w}_g}\right).$$

IA.4 Robustness checks of empirical results

We now assess the robustness of our empirical results in Section 7.3 to considering (i) the tail risk of the portfolios, (ii) the cumulative wealth derived from the portfolios, (iii) different risk-aversion coefficients, (iv) a higher level of transaction costs, (v) and a different combination of portfolios that exploits the SMV portfolio, the SGMV portfolio, and the equally weighted portfolio as in Tu and Zhou (2011). We confirm that the insights from Section 7.3 are robust to these alternative experimental settings.

IA.4.1 Value-at-Risk

Table IA.1 shows that the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ delivers lower Value-at-Risk than that of the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$, which is an additional empirical feature offered by our proposed robust shrinkage portfolio.

IA.4.2 Cumulative wealth

To gauge the economic magnitude of the outperformance delivered by the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$, Table IA.2 reports the cumulative wealth obtained by investing in the shrinkage portfolios using $\hat{\kappa}_E^\star$ and $\hat{\kappa}_R^\star$.[23] We only consider the overlapping out-of-sample period across the six datasets for comparison purposes. The table shows that the outperformance delivered by the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ is economically significant, translating into a large increase of cumulative wealth relative to the shrinkage portfolio using $\hat{\kappa}_E^\star$. In particular, the cumulative wealth increases by 255% when $T = 120$, and by 75.8% when $T = 240$, on average across the six datasets.

[23] We standardize their out-of-sample returns to have the same volatility for comparison purposes. We take as target volatility that of the market factor over the same out-of-sample period, which we download from Kenneth French's website.

IA.4.3 Different risk-aversion coefficients

Tables IA.3 and IA.4 replicate the results in Tables 2 and 3 for the case where the risk-aversion coefficient is $\gamma = 1$ and 5, respectively. For conciseness, we do not report the performance of the EW portfolio because the RTR portfolio always outperforms it, and we do not report the abysmal performance of the MV portfolio either.

For the case with $\gamma = 1$ in Table IA.3, we observe that the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ delivers a better OOS CER than the RTR portfolio, the SGMV portfolio, and the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$, both before and after transaction costs. We also see that the outperformance net of transaction costs delivered by $\hat{\kappa}_R^\star$ relative to $\hat{\kappa}_E^\star$ is larger than in the case with $\gamma = 3$. For the case with $\gamma = 5$ in Table IA.4, we observe that the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$ delivers a better OOS CER and OOS Sharpe ratio than the RTR portfolio, the SGMV portfolio, and the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$, both before and after transaction costs. Overall, the results presented in this section confirm that the insights shown in the main body of the manuscript are robust to considering different degrees of risk aversion.

IA.4.4 Higher level of transaction costs

In Table IA.5, we replicate the results in Tables 2 and 3 using a higher level of proportional transaction cost of 30 basis points in Equation (38). For conciseness, we do not report the performance of the EW portfolio because the RTR portfolio always outperforms it, and we do not report the abysmal performance of the MV portfolio either.

Our main insight in the main body of the manuscript is robust to considering a higher level of proportional transaction costs. Because the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ is more exposed to the SMV portfolio than the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$, the impact of a higher level of transaction costs on the performance of the shrinkage portfolio exploiting $\hat{\kappa}_E^\star$ is more severe than that of the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$. Comparing the SGMV portfolio and the shrinkage portfolio exploiting $\hat{\kappa}_R^\star$, we find that $\hat{\kappa}_R^\star$ continues to deliver a better performance net of transaction costs in nearly all cases. Finally, the shrinkage portfolio exploiting intensity $\hat{\kappa}_R^\star$ maintains a better net performance than that of the RTR portfolio, except for the *49IND* dataset.

IA.4.5 Combining with the equally weighted portfolio

Motivated by the finding of DeMiguel et al. (2009b) that the equally weighted (EW) portfolio often outperforms mean-variance portfolios out of sample, Tu and Zhou (2011) extend the framework introduced by Kan and Zhou (2007) and combine several estimates of the mean-variance portfolio with the EW portfolio. This section follows a similar approach and applies our methodology to the shrinkage portfolio that combines the SMV, SGMV, and EW portfolios.

Let $w_{ew} = e/N$ denote the EW portfolio. Then, the three-fund shrinkage portfolio that combines the SMV, SGMV, and EW portfolios is

$$\hat{w}^\star(\pi,\kappa) = (1-\pi)w_{ew} + \pi\hat{w}^\star(\kappa) = (1-\pi)w_{ew} + \pi((1-\kappa)\hat{w}_g + \kappa\hat{w}^\star), \tag{IA10}$$

with $\pi, \kappa \in [0,1]$. In the next proposition, we derive closed-form expressions for the OOSU mean and variance of the shrinkage portfolio $\hat{w}^\star(\pi,\kappa)$. This result allows us to compute the mean-risk OOSU and find the corresponding optimal shrinkage intensities $(\pi_R^\star, \kappa_R^\star)$. For notational simplicity, we introduce the following terms

$$\mu_{ew} = w_{ew}^\top \mu \quad \text{and} \quad \sigma_{ew}^2 = w_{ew}^\top \mathbf{\Sigma} w_{ew}, \tag{IA11}$$

for the mean return and variance of the EW portfolio.

Proposition IA.2. *Let Assumptions 1 and 2 hold. Then,*

1. *The out-of-sample utility mean of the three-fund shrinkage portfolio $\hat{w}^\star(\pi,\kappa)$ is*

$$\begin{aligned}\mathbb{E}[U(\hat{w}^\star(\pi,\kappa))] &= (1-\pi)\mu_{ew} + \pi\left(\mu_g + \frac{\kappa}{\gamma}\frac{T}{T-N-1}\psi^2\right) - \frac{\gamma}{2}\Bigg((1-\pi)^2\sigma_{ew}^2 \\ &+ \pi^2\left(\frac{T-2}{T-N-1}\sigma_g^2 + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)(T\psi^2+N-1)}{(T-N)(T-N-1)(T-N-3)}\right) \\ &+ 2\pi(1-\pi)\left(\sigma_g^2 + \frac{\kappa}{\gamma}\frac{T}{T-N-1}(\mu_{ew}-\mu_g)\right)\Bigg).\end{aligned} \tag{IA12}$$

2. The out-of-sample utility variance of the three-fund shrinkage portfolio $\hat{w}^\star(\pi,\kappa)$ *is*

$$\mathbb{V}[U(\hat{w}^\star(\pi,\kappa))] = \mathbb{V}\Big[\hat{w}^\star(\pi,\kappa)^\top\mu\Big] + \frac{\gamma^2}{4}\mathbb{V}\Big[\hat{w}^\star(\pi,\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\pi,\kappa)\Big] - \gamma\,\mathbb{C}\mathrm{ov}\Big[\hat{w}^\star(\pi,\kappa)^\top\mu, \hat{w}^\star(\pi,\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\pi,\kappa)\Big], \tag{IA13}$$

where the variance of the out-of-sample mean return is

$$\mathbb{V}\Big[\hat{w}^\star(\pi,\kappa)^\top\mu\Big] = \pi^2\left(\frac{\sigma_g^2\psi^2}{T-N-1} + \frac{\kappa^2\psi^2}{\gamma^2}\frac{2T(N+1)+T^2(T-N-3+2(T-N)\psi^2)}{(T-N)(T-N-1)^2(T-N-3)}\right), \tag{IA14}$$

the variance of the out-of-sample return variance is

$$\begin{aligned}\mathbb{V}\Big[\hat{w}^\star(\pi,\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\pi,\kappa)\Big] &= \pi^4\Bigg(\frac{2\sigma_g^4(N-1)(T-2)}{(T-N-1)^2(T-N-3)} \\ &+ \frac{4\kappa^2\sigma_g^2}{\gamma^2}\frac{T(T-2)(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} \\ &+ \frac{2\kappa^4}{\gamma^4}\frac{T^2(T-2)C(T,N,\psi^2)}{(T-N)^2(T-N-1)^2(T-N-2)(T-N-3)^2(T-N-5)(T-N-7)}\Bigg) \\ &+ 4\pi^2(1-\pi)^2\Bigg((\sigma_{ew}^2-\sigma_g^2)\left(\frac{\sigma_g^2}{T-N-1} + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)+T^2\psi^2}{(T-N)(T-N-1)(T-N-3)}\right) \\ &+ \frac{\kappa^2}{\gamma^2}\frac{T^2(T-N+1)}{(T-N)(T-N-1)^2(T-N-3)}(\mu_{ew}-\mu_g)^2\Bigg) + 8\pi^3(1-\pi)\frac{\kappa}{\gamma}(\mu_{ew}-\mu_g) \\ &\left(\sigma_g^2\frac{T(T-2)}{(T-N-1)^2(T-N-3)} + \frac{\kappa^2}{\gamma^2}\frac{T^2(T-2)(T+N-3+2T\psi^2)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)}\right),\end{aligned} \tag{IA15}$$

and the covariance between the out-of-sample mean return and variance is

$$\begin{aligned}\mathbb{C}\mathrm{ov}\Big[\hat{w}^\star(\pi,\kappa)^\top\mu, \hat{w}^\star(\pi,\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\pi,\kappa)\Big] &= 2\pi^3\frac{\kappa}{\gamma}\Bigg(\sigma_g^2\psi^2\frac{T(T-2)}{(T-N-1)^2(T-N-3)} \\ &+ \frac{\kappa^2\psi^2}{\gamma^2}\frac{T^2(T-2)(T+N-3+2T\psi^2)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)}\Bigg) + 2\pi^2(1-\pi)(\mu_{ew}-\mu_g) \\ &\left(\frac{\sigma_g^2}{T-N-1} + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)(T-N-1)+2T^2(T-N)\psi^2}{(T-N)(T-N-1)^2(T-N-3)}\right).\end{aligned} \tag{IA16}$$

Using the result in Proposition IA.2, we can find numerically the shrinkage intensities

$(\pi_E^\star, \kappa_E^\star)$ maximizing OOSU mean and the shrinkage intensities $(\pi_R^\star, \kappa_R^\star)$ maximizing our proposed mean-risk OOSU robustness metric in (27). Those shrinkage intensities depend on the following distributional parameters of stock returns: μ_g, σ_g^2, ψ^2, μ_{ew}, and σ_{ew}^2. We estimate σ_g^2 and ψ^2 as in Appendix IA.3. We estimate μ_{ew} and σ_{ew}^2 via the plug-in estimators

$$\hat{\mu}_{ew} = w_{ew}^\top \hat{\mu} \quad \text{and} \quad \hat{\sigma}_{ew}^2 = w_{ew}^\top \widehat{\mathbf{\Sigma}} w_{ew}, \tag{IA17}$$

as in Tu and Zhou (2011). Finally, we estimate μ_g as the mean return of the shrinkage portfolio that combines the EW and SGMV portfolios instead of using the plug-in estimator of μ_g, which is highly contaminated by estimation errors. We select the shrinkage intensity π of this shrinkage portfolio as the parameter π that minimizes the mean squared error of the out-of-sample mean return of the portfolio.

Proposition IA.3. *Let Assumptions 1 and 2 hold. Then, the shrinkage portfolio* $\hat{w}(\pi) = \pi w_{ew} + (1-\pi)\hat{w}_g$ *that minimizes the mean squared error* $\mathbb{E}[(\hat{w}(\pi)'\mu - \mu_g)^2]$ *is obtained for*

$$\pi = \frac{\sigma_g^2 \psi^2}{\sigma_g^2 \psi^2 + (\mu_{ew} - \mu_g)(T - N - 1)}. \tag{IA18}$$

Using Proposition IA.3, we estimate μ_g as

$$\hat{\mu}_g = \hat{w}(\hat{\pi})^\top \hat{\mu} \quad \text{with} \quad \hat{\pi} = \frac{\hat{\sigma}_g^2 \hat{\psi}_{kz}^2}{\hat{\sigma}_g^2 \hat{\psi}_{kz}^2 + (\hat{\mu}_{ew} - \hat{w}_g^\top \hat{\mu})(T - N - 1)}, \tag{IA19}$$

where $\hat{\psi}_{kz}^2$ is defined in (IA8), $\hat{\sigma}_g^2$ in (IA9), and $\hat{\mu}_{ew}$ in (IA17). Finally, using those estimators of the distributional parameters of stock returns, we can obtain the estimated shrinkage intensities $(\hat{\pi}_E^\star, \hat{\kappa}_E^\star)$ and $(\hat{\pi}_R^\star, \hat{\kappa}_R^\star)$ numerically using the results in Proposition IA.2.

In Tables IA.6, we report the out-of-sample performance of the shrinkage portfolio $\hat{w}^\star(\pi, \kappa)$ in (IA10) that combines the SMV, SGMV, and EW portfolios using the intensities $(\hat{\pi}_E^\star, \hat{\kappa}_E^\star)$ and $(\hat{\pi}_R^\star, \hat{\kappa}_R^\star)$. As in Tables 2 and 3, we estimate the portfolios with a covariance matrix that shrinks the sample covariance matrix toward the identity. The SGMV portfolio uses the shrinkage approach of Ledoit and Wolf (2004) and the two shrinkage portfolios use a shrinkage intensity calibrated via bootstrap (see Footnote 18) to optimize the same criterion used to combine the SGMV and SMV portfolios.

We observe that the results in the manuscript are robust to considering the EW portfolio in the shrinkage portfolios. Specifically, the robust shrinkage portfolio exploiting $(\hat{\pi}_R^\star, \hat{\kappa}_R^\star)$ delivers a greater OOS CER than the shrinkage portfolio maximizing OOSU mean in all cases except one before transaction costs and all cases after transaction costs. It also systematically delivers a greater Sharpe ratio. Similarly, the robust shrinkage portfolio outperforms the SGMV portfolio in most cases, before and after transaction costs.

Overall, the results presented in this section confirm that our proposed robustness measure can be applied in the construction of other investment strategies and outperform combinations of portfolios that only focus on maximizing OOSU mean as well as the individual portfolios being combined.

Table IA.1: Value-at-Risk of shrinkage portfolios

		OOS 1% VaR		OOS 5% VaR	
		$\hat{\kappa}_E^\star$	$\hat{\kappa}_R^\star$	$\hat{\kappa}_E^\star$	$\hat{\kappa}_R^\star$
Panel A: Without transaction costs					
$T = 120$	*10MOM*	0.249	0.184	0.119	0.081
	25SBTM	0.203	0.102	0.099	0.058
	25OPINV	0.207	0.125	0.094	0.062
	49IND	0.129	0.084	0.067	0.056
	16LTANOM	0.197	0.122	0.107	0.076
	46ANOM	0.186	0.157	0.117	0.091
$T = 240$	*10MOM*	0.287	0.218	0.132	0.102
	25SBTM	0.215	0.155	0.099	0.064
	25OPINV	0.203	0.136	0.103	0.071
	49IND	0.142	0.092	0.070	0.060
	16LTANOM	0.183	0.116	0.117	0.079
	46ANOM	0.612	0.408	0.247	0.212
Panel B: Net of transaction costs					
$T = 120$	*10MOM*	0.252	0.187	0.121	0.083
	25SBTM	0.212	0.105	0.110	0.060
	25OPINV	0.211	0.127	0.097	0.063
	49IND	0.131	0.084	0.074	0.056
	16LTANOM	0.200	0.122	0.114	0.076
	46ANOM	0.189	0.160	0.121	0.095
$T = 240$	*10MOM*	0.291	0.221	0.133	0.104
	25SBTM	0.218	0.158	0.102	0.068
	25OPINV	0.204	0.136	0.106	0.071
	49IND	0.143	0.093	0.071	0.060
	16LTANOM	0.186	0.117	0.120	0.080
	46ANOM	0.616	0.412	0.259	0.218

Notes. This table reports the out-of-sample 1% and 5% Value-at-Risk of the estimated shrinkage portfolio that maximizes out-of-sample utility mean ($\hat{\kappa}_E^\star$) and the estimated shrinkage portfolio that maximizes the portfolio robustness measure in Section 6 ($\hat{\kappa}_R^\star$) for the six datasets of monthly excess returns described in Section 7.1. We estimate the shrinkage portfolios using sample sizes of $T = 120$ and $T = 240$ monthly observations, a risk-aversion coefficient of $\gamma = 3$, and a coefficient $\lambda = 2$ for the shrinkage portfolio that maximizes the proposed robustness measure. The out-of-sample Value-at-Risk is computed as the negative 1st and 5th percentiles of the out-of-sample monthly excess returns. Panel A considers the case without transaction costs, and Panel B considers the case with proportional transaction costs of 20 basis points.

Table IA.2: Cumulative wealth net of transaction costs of shrinkage portfolios

	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$	% increase
Panel A: $T = 120$ (July 1983 − December 2013)			
10MOM	88.5	140	59%
25SBTM	149	740	396%
25OPINV	34.6	166	381%
49IND	5.44	24.2	346%
16LTANOM	27.5	119	332%
46ANOM	49,686	56,731	14%
Panel B: $T = 240$ (July 1993 − December 2013)			
10MOM	5.86	7.70	31%
25SBTM	18.3	23.4	28%
25OPINV	9.83	17.2	75%
49IND	2.18	4.70	116%
16LTANOM	8.84	19.0	115%
46ANOM	64.8	123	90%

Notes. This table reports the cumulative wealth net of proportional transaction costs of 20 basis points of the estimated shrinkage portfolios. We consider the estimated shrinkage portfolio that maximizes out-of-sample utility mean ($\hat{\kappa}^{\star}_{E}$), and the estimated shrinkage portfolio that maximizes the portfolio robustness measure in Section 6 ($\hat{\kappa}^{\star}_{R}$). We report the cumulative wealth of the shrinkage portfolios for the six datasets of monthly excess returns described in Section 7.1. The out-of-sample returns include the risk-free rate and are standardized to have the same volatility as that of the market factor during the same time period. We only consider the overlapping out-of-sample period across the six datasets, which spans July 1983 through December 2013 when the portfolios are estimated with a sample size of $T = 120$ (Panel A) and July 1993 through December 2013 when the portfolios are estimated with a sample size of $T = 240$ (Panel B). We use a risk-aversion coefficient of $\gamma = 3$ and a coefficient $\lambda = 2$ for the shrinkage portfolio that maximizes the proposed robustness measure.

Table IA.3: Out-of-sample performance with lower risk aversion

		$T = 120$				$T = 240$			
		RTR	SGMV	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$	RTR	SGMV	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$
10MOM	Gross OOS CER	0.077	0.082	0.238	0.286	0.079	0.088	0.274	0.323
	Net OOS CER	0.075	0.078	0.150	0.243	0.078	0.084	0.218	0.289
	Gross OOS SR	0.558	0.641	0.745	0.770	0.607	0.706	0.784	0.804
	Net OOS SR	0.547	0.611	0.654	0.702	0.600	0.682	0.727	0.761
	Average $\hat{\kappa}$	/	0	0.395	0.273	/	0	0.546	0.454
25SBTM	Gross OOS CER	0.087	0.088	0.231	0.231	0.086	0.096	0.253	0.271
	Net OOS CER	0.086	0.079	0.052	0.177	0.085	0.089	0.133	0.199
	Gross OOS SR	0.561	0.748	0.693	0.866	0.612	0.833	0.730	0.802
	Net OOS SR	0.554	0.679	0.464	0.708	0.606	0.778	0.589	0.653
	Average $\hat{\kappa}$	/	0	0.214	0.131	/	0	0.364	0.280
25OPINV	Gross OOS CER	0.079	0.097	0.107	0.154	0.081	0.119	0.191	0.209
	Net OOS CER	0.077	0.088	-0.008	0.107	0.080	0.114	0.119	0.167
	Gross OOS SR	0.586	0.794	0.495	0.656	0.633	0.976	0.619	0.745
	Net OOS SR	0.575	0.729	0.318	0.504	0.627	0.935	0.505	0.635
	Average $\hat{\kappa}$	/	0	0.191	0.104	/	0	0.320	0.221
49IND	Gross OOS CER	0.075	0.067	0.068	0.070	0.073	0.058	0.020	0.056
	Net OOS CER	0.074	0.058	0.011	0.051	0.072	0.051	-0.017	0.031
	Gross OOS SR	0.565	0.624	0.395	0.550	0.592	0.557	0.234	0.391
	Net OOS SR	0.554	0.546	0.172	0.423	0.585	0.501	0.131	0.256
	Average $\hat{\kappa}$	/	0	0.043	0.016	/	0	0.087	0.041
16LTANOM	Gross OOS CER	0.077	0.084	0.191	0.206	0.071	0.111	0.341	0.351
	Net OOS CER	0.075	0.078	0.046	0.151	0.070	0.107	0.235	0.295
	Gross OOS SR	0.516	0.695	0.640	0.766	0.520	0.942	0.836	0.907
	Net OOS SR	0.505	0.649	0.452	0.611	0.513	0.904	0.724	0.808
	Average $\hat{\kappa}$	/	0	0.309	0.201	/	0	0.522	0.443
46ANOM	Gross OOS CER	0.079	0.067	2.378	2.179	0.083	0.111	0.240	0.850
	Net OOS CER	0.077	0.054	2.177	2.011	0.082	0.101	-0.179	0.736
	Gross OOS SR	0.566	0.643	2.364	2.372	0.600	1.037	1.474	1.565
	Net OOS SR	0.555	0.532	2.249	2.265	0.593	0.945	1.228	1.491
	Average $\hat{\kappa}$	/	0	0.250	0.205	/	0	0.511	0.471

Notes. This table reports the out-of-sample performance of four portfolio strategies described in Section 7.3 for the six datasets of monthly excess returns described in Section 7.1. Each estimated portfolio is constructed using a sample size of $T = 120$ and 240 monthly observations. The mean-variance portfolios consider a risk-aversion coefficient of $\gamma = 1$. The table reports the annualized out-of-sample certainty-equivalent return (OOS CER) and the annualized out-of-sample Sharpe ratio (OOS SR), and for both criteria we report the gross performance and the performance net of proportional transaction costs of 20 basis points. We also report the average estimated shrinkage intensity $\hat{\kappa}$ over time, except for the RTR portfolio that does not combine the SMV and SGMV portfolios.

Table IA.4: Out-of-sample performance with higher risk aversion

		$T=120$				$T=240$			
		RTR	SGMV	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$	RTR	SGMV	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$
10MOM	Gross OOS CER	0.025	0.040	0.072	0.084	0.035	0.050	0.091	0.101
	Net OOS CER	0.023	0.036	0.051	0.072	0.034	0.047	0.078	0.090
	Gross OOS SR	0.558	0.641	0.915	0.916	0.607	0.706	0.985	1.005
	Net OOS SR	0.547	0.611	0.829	0.851	0.600	0.682	0.931	0.952
	Average $\hat{\kappa}$	/	0	0.395	0.312	/	0	0.546	0.483
25SBTM	Gross OOS CER	0.018	0.055	0.089	0.086	0.034	0.065	0.102	0.109
	Net OOS CER	0.016	0.046	0.046	0.067	0.033	0.058	0.074	0.083
	Gross OOS SR	0.561	0.748	0.949	0.958	0.612	0.833	1.012	1.077
	Net OOS SR	0.554	0.679	0.745	0.828	0.606	0.778	0.881	0.920
	Average $\hat{\kappa}$	/	0	0.214	0.155	/	0	0.364	0.298
25OPINV	Gross OOS CER	0.031	0.062	0.059	0.075	0.039	0.085	0.097	0.105
	Net OOS CER	0.029	0.053	0.031	0.060	0.038	0.080	0.081	0.093
	Gross OOS SR	0.586	0.794	0.787	0.882	0.633	0.976	0.985	1.070
	Net OOS SR	0.575	0.729	0.638	0.777	0.627	0.935	0.899	0.992
	Average $\hat{\kappa}$	/	0	0.191	0.132	/	0	0.320	0.244
49IND	Gross OOS CER	0.028	0.039	0.029	0.044	0.034	0.031	0.016	0.036
	Net OOS CER	0.026	0.030	0.009	0.034	0.033	0.025	0.003	0.029
	Gross OOS SR	0.565	0.624	0.553	0.668	0.592	0.557	0.459	0.598
	Net OOS SR	0.554	0.546	0.401	0.582	0.585	0.501	0.359	0.540
	Average $\hat{\kappa}$	/	0	0.043	0.029	/	0	0.087	0.061
16LTANOM	Gross OOS CER	0.011	0.048	0.064	0.076	0.018	0.079	0.125	0.127
	Net OOS CER	0.009	0.042	0.029	0.059	0.017	0.075	0.103	0.114
	Gross OOS SR	0.516	0.695	0.829	0.878	0.520	0.942	1.118	1.183
	Net OOS SR	0.505	0.649	0.658	0.766	0.513	0.904	1.021	1.104
	Average $\hat{\kappa}$	/	0	0.309	0.232	/	0	0.522	0.455
46ANOM	Gross OOS CER	0.026	0.041	0.513	0.472	0.032	0.085	0.109	0.237
	Net OOS CER	0.024	0.028	0.474	0.439	0.031	0.075	0.002	0.203
	Gross OOS SR	0.566	0.643	2.445	2.435	0.600	1.037	1.596	1.717
	Net OOS SR	0.555	0.532	2.327	2.324	0.593	0.945	1.377	1.641
	Average $\hat{\kappa}$	/	0	0.250	0.208	/	0	0.511	0.472

Notes. This table reports the out-of-sample performance of four portfolio strategies described in Section 7.3 for the six datasets of monthly excess returns described in Section 7.1. Each estimated portfolio is constructed using a sample size of $T = 120$ and 240 monthly observations. The mean-variance portfolios consider a risk-aversion coefficient of $\gamma = 5$. The table reports the annualized out-of-sample certainty-equivalent return (OOS CER) and the annualized out-of-sample Sharpe ratio (OOS SR), and for both criteria we report the gross performance and the performance net of proportional transaction costs of 20 basis points. We also report the average estimated shrinkage intensity $\hat{\kappa}$ over time, except for the RTR portfolio that does not combine the SMV and SGMV portfolios.

Table IA.5: Out-of-sample performance with higher transaction costs

		$T = 120$				$T = 240$			
		RTR	SGMV	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$	RTR	SGMV	$\hat{\kappa}^{\star}_{E}$	$\hat{\kappa}^{\star}_{R}$
10MOM	Gross OOS CER	0.051	0.061	0.114	0.131	0.057	0.069	0.135	0.152
	Net OOS CER	0.049	0.055	0.065	0.105	0.056	0.064	0.104	0.131
	Gross OOS SR	0.558	0.641	0.864	0.898	0.607	0.706	0.917	0.958
	Net OOS SR	0.542	0.597	0.727	0.793	0.597	0.670	0.834	0.888
	Average $\hat{\kappa}$	/	0	0.395	0.296	/	0	0.546	0.471
25SBTM	Gross OOS CER	0.052	0.072	0.122	0.123	0.060	0.081	0.139	0.146
	Net OOS CER	0.050	0.058	0.022	0.086	0.058	0.070	0.073	0.096
	Gross OOS SR	0.561	0.748	0.858	0.995	0.612	0.833	0.914	1.014
	Net OOS SR	0.550	0.644	0.523	0.768	0.603	0.750	0.705	0.773
	Average $\hat{\kappa}$	/	0	0.214	0.145	/	0	0.364	0.289
25OPINV	Gross OOS CER	0.055	0.079	0.080	0.101	0.060	0.102	0.124	0.133
	Net OOS CER	0.052	0.066	0.018	0.073	0.059	0.094	0.087	0.109
	Gross OOS SR	0.586	0.794	0.699	0.866	0.633	0.976	0.872	1.004
	Net OOS SR	0.569	0.696	0.458	0.696	0.623	0.915	0.722	0.872
	Average $\hat{\kappa}$	/	0	0.191	0.120	/	0	0.320	0.232
49IND	Gross OOS CER	0.052	0.053	0.043	0.059	0.054	0.044	0.024	0.046
	Net OOS CER	0.049	0.039	-0.005	0.041	0.052	0.035	0.001	0.031
	Gross OOS SR	0.565	0.624	0.510	0.661	0.592	0.557	0.392	0.558
	Net OOS SR	0.549	0.507	0.194	0.519	0.581	0.473	0.250	0.437
	Average $\hat{\kappa}$	/	0	0.043	0.024	/	0	0.087	0.054
16LTANOM	Gross OOS CER	0.044	0.066	0.099	0.110	0.045	0.095	0.172	0.175
	Net OOS CER	0.041	0.057	0.020	0.077	0.043	0.088	0.118	0.149
	Gross OOS SR	0.516	0.695	0.779	0.893	0.520	0.942	1.016	1.130
	Net OOS SR	0.499	0.626	0.508	0.703	0.509	0.886	0.856	1.011
	Average $\hat{\kappa}$	/	0	0.309	0.218	/	0	0.522	0.449
46ANOM	Gross OOS CER	0.052	0.054	0.832	0.765	0.057	0.098	0.147	0.347
	Net OOS CER	0.050	0.035	0.738	0.687	0.056	0.083	0.003	0.268
	Gross OOS SR	0.566	0.643	2.433	2.442	0.600	1.037	1.543	1.648
	Net OOS SR	0.550	0.476	2.259	2.278	0.589	0.900	1.347	1.534
	Average $\hat{\kappa}$	/	0	0.250	0.207	/	0	0.511	0.471

Notes. This table reports the out-of-sample performance of four portfolio strategies described in Section 7.3 for the six datasets of monthly excess returns described in Section 7.1. Each estimated portfolio is constructed using a sample size of $T = 120$ and 240 monthly observations. The mean-variance portfolios consider a risk-aversion coefficient of $\gamma = 3$. The table reports the annualized out-of-sample certainty-equivalent return (OOS CER) and the annualized out-of-sample Sharpe ratio (OOS SR), and for both criteria we report the gross performance and the performance net of proportional transaction costs of 30 basis points. We also report the average estimated shrinkage intensity $\hat{\kappa}$ over time, except for the RTR portfolio that does not combine the SMV and SGMV portfolios.

Table IA.6: Out-of-sample performance exploiting the equally weighted portfolio

		$T = 120$				$T = 240$			
		EW	SGMV	$\hat{\kappa}^\star_E$	$\hat{\kappa}^\star_R$	EW	SGMV	$\hat{\kappa}^\star_E$	$\hat{\kappa}^\star_R$
10MOM	Gross OOS CER	0.034	0.061	0.110	0.120	0.039	0.069	0.132	0.147
	Net OOS CER	0.033	0.057	0.079	0.102	0.038	0.065	0.111	0.133
	Gross OOS SR	0.455	0.641	0.851	0.854	0.484	0.706	0.909	0.942
	Net OOS SR	0.449	0.611	0.764	0.784	0.478	0.682	0.851	0.894
	Average $\hat{\kappa}$	/	0	0.692	0.674	/	0	0.784	0.781
	Average $\hat{\pi}$	0	/	0.637	0.473	0	/	0.724	0.629
25SBTM	Gross OOS CER	0.045	0.072	0.122	0.120	0.053	0.081	0.142	0.151
	Net OOS CER	0.044	0.063	0.059	0.090	0.052	0.074	0.099	0.103
	Gross OOS SR	0.522	0.748	0.858	0.945	0.567	0.833	0.923	1.026
	Net OOS SR	0.516	0.679	0.649	0.775	0.561	0.778	0.788	0.803
	Average $\hat{\kappa}$	/	0	0.648	0.628	/	0	0.663	0.658
	Average $\hat{\pi}$	0	/	0.427	0.312	0	/	0.600	0.471
25OPINV	Gross OOS CER	0.044	0.079	0.074	0.090	0.050	0.102	0.119	0.126
	Net OOS CER	0.043	0.071	0.032	0.069	0.049	0.097	0.094	0.111
	Gross OOS SR	0.515	0.794	0.671	0.802	0.556	0.976	0.851	0.969
	Net OOS SR	0.509	0.729	0.505	0.673	0.550	0.935	0.749	0.883
	Average $\hat{\kappa}$	/	0	0.820	0.786	/	0	0.867	0.866
	Average $\hat{\pi}$	0	/	0.272	0.170	0	/	0.381	0.265
49IND	Gross OOS CER	0.050	0.053	0.038	0.049	0.052	0.044	0.037	0.047
	Net OOS CER	0.049	0.044	0.013	0.039	0.051	0.038	0.024	0.038
	Gross OOS SR	0.552	0.624	0.481	0.560	0.569	0.557	0.471	0.547
	Net OOS SR	0.544	0.546	0.336	0.491	0.562	0.501	0.393	0.485
	Average $\hat{\kappa}$	/	0	0.431	0.365	/	0	0.418	0.393
	Average $\hat{\pi}$	0	/	0.232	0.131	0	/	0.249	0.133
16LTANOM	Gross OOS CER	0.027	0.066	0.094	0.109	0.028	0.095	0.165	0.167
	Net OOS CER	0.026	0.060	0.043	0.086	0.027	0.091	0.128	0.150
	Gross OOS SR	0.419	0.695	0.762	0.883	0.422	0.942	0.994	1.103
	Net OOS SR	0.413	0.649	0.588	0.750	0.416	0.904	0.886	1.020
	Average $\hat{\kappa}$	/	0	0.783	0.784	/	0	0.839	0.859
	Average $\hat{\pi}$	0	/	0.443	0.301	0	/	0.672	0.573
46ANOM	Gross OOS CER	0.016	0.054	0.302	0.765	0.022	0.098	-1.007	0.374
	Net OOS CER	0.015	0.041	0.126	0.713	0.021	0.088	-1.067	0.320
	Gross OOS SR	0.358	0.643	1.999	2.436	0.399	1.037	1.415	1.665
	Net OOS SR	0.353	0.532	1.746	2.328	0.393	0.945	1.279	1.587
	Average $\hat{\kappa}$	/	0	0.727	0.713	/	0	0.799	0.801
	Average $\hat{\pi}$	0	/	0.494	0.454	0	/	0.698	0.651

Notes. This table reports the out-of-sample performance of the shrinkage portfolios that combine the SMV, SGMV, and equally weighted portfolios introduced in Appendix IA.4.5 for the six datasets of monthly excess returns described in Section 7.1. Each estimated portfolio is constructed using a sample size of $T = 120$ and 240 monthly observations. The mean-variance portfolios consider a risk-aversion coefficient of $\gamma = 3$. The table reports the annualized out-of-sample certainty-equivalent return (OOS CER) and the annualized out-of-sample Sharpe ratio (OOS SR), and for both criteria we report the gross performance and the performance net of proportional transaction costs of 20 basis points. We also report the average estimated shrinkage intensities $\hat{\pi}$ and $\hat{\kappa}$ over time, which determine the combination of the three portfolios in Equation (IA10).

IA.5 Proofs of all results

Throughout the proofs presented in this section, we use the fact that the shrinkage portfolio $\hat{w}^\star(\kappa)$ in (11) can be rewritten as

$$\hat{w}^\star(\kappa) = \hat{w}_g + \frac{\kappa}{\gamma}\hat{w}_z. \tag{IA20}$$

Proof of Proposition 1

Kan et al. (2021b, Proposition 1) show that the finite-sample distribution of out-of-sample mean return and variance, and thus of OOSU, of the shrinkage portfolio $\hat{w}^\star(\kappa)$ is a function of 12 univariate independent random variables. Six of those random variables are normally distributed, and the remaining six are the following chi-square distributions: $u_0 \sim \chi^2_{N-2}$, $s_1 \sim \chi^2_{N-4}$, $s_2 \sim \chi^2_{N-3}$, $v_2 \sim \chi^2_{T-N+1}$, $w_1 \sim \chi^2_{T-N+3}$, and $w_2 \sim \chi^2_{T-N+2}$. In the high-dimensional asymptotic setting where $N, T \to \infty$ and $N/T \to \rho \in [0, 1)$, these six chi-square random variables are normally distributed. Specifically, from the proof of Kan et al. (2021a, Proposition 4), we have that

$$\begin{aligned}
\sqrt{T}(u_0/T - \rho) &\xrightarrow{d} \mathcal{N}(0, 2\rho), \\
\sqrt{T}(s_1/T - \rho) &\xrightarrow{d} \mathcal{N}(0, 2\rho), \\
\sqrt{T}(s_2/T - \rho) &\xrightarrow{d} \mathcal{N}(0, 2\rho), \\
\sqrt{T}(v_2/T - (1-\rho)) &\xrightarrow{d} \mathcal{N}(0, 2(1-\rho)), \\
\sqrt{T}(w_1/T - (1-\rho)) &\xrightarrow{d} \mathcal{N}(0, 2(1-\rho)), \\
\sqrt{T}(w_2/T - (1-\rho)) &\xrightarrow{d} \mathcal{N}(0, 2(1-\rho)).
\end{aligned}$$

Therefore, $\sqrt{T}U(\hat{w}^\star(\kappa))$ is asymptotically a function of 12 independent univariate Gaussian random variables. Using the delta method, it then follows that the asymptotic distribution of $\sqrt{T}U(\hat{w}^\star(\kappa))$ is Gaussian as well.

To find the asymptotic mean and variance, $u(\kappa, \rho)$ and $v(\kappa, \rho)$, we proceed as follows. First, we take the finite-sample expressions for OOSU mean and variance, $\mathbb{E}[U(\hat{w}^\star(\kappa))]$ and

$\mathbb{V}[U(\hat{w}^\star(\kappa))]$, which are available in Propositions 2 and 3, respectively. Second, we only keep the dominant terms in N and T. Third, in the resulting expressions, we set $N = \rho T$. Following these three steps, it is straightforward to show that $\mathbb{E}[U(\hat{w}^\star(\kappa))] \to u(\kappa, \rho)$ and $T\mathbb{V}[U(\hat{w}^\star(\kappa))] \to v(\kappa, \rho)$ as $N, T \to \infty$ with $N/T \to \rho \in [0, 1)$, which proves the proposition.

Proof of Proposition 2

The proof of this proposition is in Kan et al. (2021b).

Proof of Lemma 1

Equation (18) is directly obtained from the definition of OOSU in (12) and the formula for the variance of a sum of two correlated random variables.

Proof of Proposition 3

Kan et al. (2021b, Proposition 1) derive a stochastic representation for the out-of-sample mean return and variance of the shrinkage portfolio $\hat{w}(\kappa)$ that combines the SMV and SGMV portfolios, i.e., for the two random variables $\hat{w}^\star(\kappa)^\top \mu$ and $\hat{w}^\star(\kappa)^\top \boldsymbol{\Sigma} \hat{w}^\star(\kappa)$. Using this result, we can find analytical expressions for the three terms composing the OOSU variance in (18). Specifically, the variance of the out-of-sample mean return is

$$\mathbb{V}\Big[\hat{w}^\star(\kappa)^\top \mu\Big] = \frac{\sigma_g^2 \psi^2}{T-N-1} + \frac{\kappa^2 \psi^2}{\gamma^2} \frac{2T(N+1) + T^2(T-N-3+2(T-N)\psi^2)}{(T-N)(T-N-1)^2(T-N-3)}, \tag{IA21}$$

the variance of the out-of-sample return variance is

$$\begin{aligned}\mathbb{V}\Big[\hat{w}^\star(\kappa)^\top \boldsymbol{\Sigma} \hat{w}^\star(\kappa)\Big] &= \frac{2\sigma_g^4 (N-1)(T-2)}{(T-N-1)^2(T-N-3)} \\ &+ \frac{4\kappa^2\sigma_g^2}{\gamma^2} \frac{T(T-2)(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} \\ &+ \frac{2\kappa^4}{\gamma^4} \frac{T^2(T-2)C(T,N,\psi^2)}{(T-N)^2(T-N-1)^2(T-N-2)(T-N-3)^2(T-N-5)(T-N-7)},\end{aligned} \tag{IA22}$$

where $C(T, N, \psi^2)$ is defined in Proposition 3, and the covariance between the out-of-sample mean return and variance is

$$\begin{aligned}\mathbb{C}\text{ov}\Big[\hat{w}^\star(\kappa)^\top\mu, \hat{w}^\star(\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big] &= \frac{2\kappa\sigma_g^2\psi^2}{\gamma}\frac{T(T-2)}{(T-N-1)^2(T-N-3)} \\ &+ \frac{2\kappa^3\psi^2}{\gamma^3}\frac{T^2(T-2)(T+N-3+2T\psi^2)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)}.\end{aligned} \tag{IA23}$$

We find the final expression for the OOSU variance in Proposition 3 by plugging (IA21)–(IA23) into (18).

Proof of Corollary 1

First, we prove that the shrinkage intensity that minimizes OOSU variance, $\kappa_V^\star$, is strictly positive. The derivative of the OOSU variance in (19) with respect to κ, evaluated at $\kappa = 0$, is

$$\left.\frac{\partial\mathbb{V}[U(\hat{w}^\star(\kappa))]}{\partial\kappa}\right|_{\kappa=0} = a_4. \tag{IA24}$$

Now, observe that a_4 in Equation (25) is strictly negative when $\psi^2 > 0$ because $\sigma_g^2 > 0$. Therefore, provided that $\psi^2 > 0$, it is always optimal to choose a shrinkage intensity $\kappa > 0$ to minimize OOSU variance.

Second, we prove that $\kappa_V^\star < 1$, which follows from Proposition 7 where we show that $\kappa_V^\star \leq \kappa_E^\star$. Indeed, because $\kappa_E^\star < 1$ as long as the sample size T is finite, it follows that $\kappa_V^\star < 1$.

Proof of Proposition 4

Parameters T and N. From the closed-form expression of $\mathbb{V}[U(\hat{w}^\star(\kappa))]$ in Proposition 3, it is straightforward to see that it is decreasing in T and increasing in N. In particular, it is easy to check that $\mathbb{V}[U(\hat{w}^\star(\kappa))] \to 0$ as $T \to \infty$ for any shrinkage intensity κ.

Parameter σ_g^2. The derivative of the OOSU variance with respect to σ_g^2 is

$$\frac{\partial\mathbb{V}[U(\hat{w}^\star(\kappa))]}{\partial\sigma_g^2} = \frac{\psi^2}{T-N-1} + \sigma_g^2\gamma^2\frac{(N-1)(T-2)}{(T-N-1)^2(T-N-3)}$$

$$+\kappa^2 \frac{T(T-2)(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} - 2\kappa\psi^2 \frac{T(T-2)}{(T-N-1)^2(T-N-3)}. \tag{IA25}$$

The objective is to show that the derivative in (IA25) is always positive. Notice that it increases with σ_g^2, and thus it suffices to show that it is always positive for the case $\sigma_g^2 = 0$. That is, we need to show that

$$\begin{aligned} &\frac{\psi^2}{T-N-1} + \kappa^2 \frac{T(T-2)(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} \\ &- 2\kappa\psi^2 \frac{T(T-2)}{(T-N-1)^2(T-N-3)} \geq 0. \end{aligned} \tag{IA26}$$

Notice that the left-hand side of (IA26) is a second-degree polynomial in κ. Because the coefficient in front of κ^2 is positive, we can prove that inequality (IA26) holds by showing that the polynomial discriminant is always negative. That is, after some simplifications,

$$\psi^2 \frac{T(T-2)}{(T-N-1)(T-N-3)} \leq \frac{(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-5)}. \tag{IA27}$$

Notice that the right-hand side of inequality (IA27) is of the form $a + b\psi^2$ with $a > 0$. Therefore, we can prove the inequality by showing that the coefficient in front of ψ^2 on the right-hand side is larger than that in front of ψ^2 on the left-hand side. This is equivalent to showing that

$$\frac{T(T+N-3)}{(T-N)(T-N-5)} \geq \frac{T(T-2)}{(T-N-1)(T-N-3)}, \tag{IA28}$$

which holds under Assumption 1.

Parameter ψ^2. The derivative of the OOSU variance with respect to ψ^2 is

$$\begin{aligned} \frac{\partial \mathbb{V}[U(\hat{w}^\star(\kappa))]}{\partial \psi^2} &= \frac{\sigma_g^2}{T-N-1} \\ &+ \frac{\kappa^4}{2\gamma^2} \frac{T^2(T-2)\frac{\partial C(T,N,\psi^2)}{\partial \psi^2}}{(T-N)^2(T-N-1)^2(T-N-2)(T-N-3)^2(T-N-5)(T-N-7)} \end{aligned}$$

$$
\begin{aligned}
&- \frac{2\kappa^3}{\gamma^2} \frac{4\psi^2 T^3(T-2) + T^2(T-2)(T+N-3)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} \\
&+ \frac{\kappa^2}{\gamma^2} \frac{2T(N+1) + T^2(T-N-3) + 4T^2(T-N)\psi^2}{(T-N)(T-N-1)^2(T-N-3)} \\
&+ \kappa^2\sigma_g^2 \frac{T^2(T-2)(T+N-3)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} \\
&- 2\kappa\sigma_g^2 \frac{T(T-2)}{(T-N-1)^2(T-N-3)}.
\end{aligned} \tag{IA29}
$$

First, we show that the derivative (IA29) is increasing in σ_g^2 and thus that it suffices to show that it is positive for $\sigma_g^2 = 0$. We have

$$
\begin{aligned}
\frac{\partial}{\partial \sigma_g^2}\left(\frac{\partial \mathbb{V}[U(\hat{w}^\star(\kappa))]}{\partial \psi^2}\right) = \frac{1}{T-N-1} &+ \kappa^2 \frac{T^2(T-2)(T+N-3)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} \\
&- 2\kappa \frac{T(T-2)}{(T-N-1)^2(T-N-3)}.
\end{aligned} \tag{IA30}
$$

Following a similar strategy to the case with σ_g^2 as a parameter, the derivative (IA30) is always positive if the polynomial discriminant is negative. This amounts to showing, after some simplifications, that

$$
\frac{(T+N-3)(T-N-1)(T-N-3)}{(T-2)(T-N)(T-N-5)} \geq 1,
$$

which holds under Assumption 1. Therefore, we can now prove the result of the proposition by showing that the derivative in (IA29) is positive for $\sigma_g^2 = 0$. That is, after some simplifications,

$$
\begin{aligned}
&\frac{\kappa^2}{2} \frac{T^2(T-2)\frac{\partial C(T,N,\psi^2)}{\partial \psi^2}}{(T-N)(T-N-2)(T-N-3)(T-N-7)} \\
&- 2\kappa\Big(4T^3(T-2)\psi^2 + T^2(T-2)(T+N-3)\Big) \\
&+ (T-N-5)\Big(2T(N+1) + T^2(T-N-3) + 4T^2(T-N)\psi^2\Big) \geq 0.
\end{aligned} \tag{IA31}
$$

We find that the derivative of (IA31) with respect to ψ^2 is positive if

$$
\frac{\partial^2 C(T,N,\psi^2)}{\partial(\psi^2)^2} \geq \frac{8T^2(T-2)(T-N-2)(T-N-3)(T-N-7)}{(T-N-5)}, \tag{IA32}
$$

where $\frac{\partial^2 C(T,N,\psi^2)}{\partial(\psi^2)^2} = 2T^2(N^3+2N^2T-6N^2-7NT^2+40NT-53N+4T^3-34T^2+88T-70)$, and inequality (IA32) holds under Assumption 1. Therefore, we can now prove the result of the proposition by showing that the derivative in (IA31) is positive for $\psi^2 = 0$. That is,

$$\frac{\kappa^2}{2}\frac{T^2(T-2)\frac{\partial C(T,N,\psi^2)}{\partial\psi^2}\Big|_{\psi^2=0}}{(T-N)(T-N-2)(T-N-3)(T-N-7)} - 2\kappa T^2(T-2)(T+N-3)$$
$$+ (T-N-5)(2T(N+1)+T^2(T-N-3)) \geq 0. \tag{IA33}$$

As usual, we prove inequality (IA33) by showing that the polynomial discriminant is negative, which amounts to showing, after some simplifications, that

$$\frac{\partial C(T,N,\psi^2)}{\partial\psi^2}\Big|_{\psi^2=0} \geq \frac{2T(T-2)(T-N)(T-N-2)(T-N-3)(T+N-3)^2}{(T-N-5)(2(N+1)+T(T-N-3))}, \tag{IA34}$$

which holds under Assumption 1, thus concluding the proof for parameter ψ^2.

Parameter κ. To prove the result we need to show that the derivative of the OOSU variance with respect to κ is positive if $\kappa \geq \kappa_E^\star$. That is,

$$4a_1\kappa^3 + 3a_2\kappa^2 + 2a_3\kappa + a_4 \geq 0 \tag{IA35}$$

if $\kappa \geq \kappa_E^\star$. As we show below, the derivative in (IA35) decreases with γ when $\kappa \geq \kappa_E^\star$. Therefore, we can derive a sufficient condition on the value of κ for which inequality (IA35) holds by considering the case $\gamma \to \infty$. In that case, the condition in (IA35) becomes

$$2\kappa\sigma_g^2\frac{T(T-2)(T+N-3)(T\psi^2+N-1)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)} - 2\sigma_g^2\psi^2\frac{T(T-2)}{(T-N-1)^2(T-N-3)} \geq 0,$$

which after isolating κ reduces to the sufficient condition

$$\kappa \geq \kappa_E^\star\frac{(T-2)(T-N-5)}{(T+N-3)(T-N-3)}, \tag{IA36}$$

which is satisfied when $\kappa \geq \kappa_E^\star$ because the right-hand side of (IA36) is smaller than $\kappa_E^\star$ under Assumption 1.

The only step missing now is to show that the left-hand side of (IA35) is decreasing in γ when $\kappa \geq \kappa_E^\star$. To prove this result, it is useful to introduce the notation

$$\begin{aligned}
\overline{a}_1 &= a_1\gamma^2,\\
\overline{a}_2 &= a_2\gamma^2,\\
\overline{a}_{3,1} &= \psi^2 \frac{2T(N+1) + T^2(T-N-3+2(T-N)\psi^2)}{(T-N)(T-N-1)^2(T-N-3)},
\end{aligned}$$

which are all independent of γ. Now, it is straightforward to show that the left-hand side of (IA35) is decreasing in γ when $\kappa \geq \kappa_E^\star$ if

$$4\overline{a}_1\kappa^2 + 3\overline{a}_2\kappa + 2\overline{a}_{3,1} \geq 0 \tag{IA37}$$

when $\kappa \geq \kappa_E^\star$. Since $\overline{a}_1 \geq 0$, inequality (IA37) holds for all κ if the polynomial discriminant is negative. Otherwise, if the discriminant is positive, we need to show that the maximum of the two real roots to the polynomial in (IA37) is smaller than $\kappa_E^\star$. That is,

$$\frac{-3\overline{a}_2 + \sqrt{9\overline{a}_2^2 - 32\overline{a}_1\overline{a}_{3,1}}}{8\overline{a}_1} \leq \kappa_E^\star. \tag{IA38}$$

After some simplifications, proving inequality (IA38) is equivalent to showing that

$$4\overline{a}_1(\kappa_E^\star)^2 + 3\overline{a}_2\kappa_E^\star + 2\overline{a}_{3,1} \geq 0. \tag{IA39}$$

We can reformulate inequality (IA39) as

$$\begin{aligned}
&\frac{\psi^2 C(T,N,\psi^2)}{(T-2)(T-N-2)(T-N-5)(T-N-7)}\\
&- \frac{3\psi^2(T\psi^2+N-1)(T+N-3+2T\psi^2)}{T-N-5}\\
&+ \frac{2T(N+1)+T^2(T-N-3+2(T-N)\psi^2)}{(T-N)(T-N-3)}\left(\psi^2 + \frac{N-1}{T}\right)^2 \geq 0.
\end{aligned} \tag{IA40}$$

Notice that inequality (IA40) holds when $\psi^2 = 0$. Therefore, we can prove (IA40) by showing that the derivative of the left-hand side with respect to ψ^2 is positive. That is,

$$
\begin{aligned}
&\frac{1}{(T-2)(T-N-2)(T-N-5)(T-N-7)}\Big[(4T\psi^2+N-1)(N^4+N^3T-3N^3\\
&-4N^2T^2+22N^2T-31N^2+NT^3-7NT^2+13NT-5N+T^4-12T^3+53T^2-100T\\
&+70)+3T^2\psi^4(N^3+2N^2T-6N^2-7NT^2+40NT-53N+4T^3-34T^2+88T-70)\Big]\\
&-\frac{3T}{T-N-5}\Big[(T+N-3)\Big(2\psi^2+\frac{N-1}{T}\Big)+2T\Big(3\psi^4+2\psi^2\frac{N-1}{T}\Big)\Big]\\
&+\frac{2T}{(T-N)(T-N-3)}\Big[(2(N+1)+T(T-N-3))\Big(\psi^2+\frac{N-1}{T}\Big)+T(T-N)\\
&\Big(3\psi^4+4\psi^2\frac{N-1}{T}+\Big(\frac{N-1}{T}\Big)^2\Big)\Big]\geq 0. \qquad \text{(IA41)}
\end{aligned}
$$

One can check that inequality (IA41) holds when $\psi^2=0$ under Assumption 1. Therefore, we can prove (IA41) by showing as before that the derivative of the left-hand side with respect to ψ^2 is positive. That is,

$$
\begin{aligned}
&\frac{2T}{(T-2)(T-N-2)(T-N-5)(T-N-7)}\Big[2(N^4+N^3T-3N^3-4N^2T^2+22N^2T\\
&-31N^2+NT^3-7NT^2+13NT-5N+T^4-12T^3+53T^2-100T+70)+3T\psi^2(N^3\\
&+2N^2T-6N^2-7NT^2+40NT-53N+4T^3-34T^2+88T-70)\Big]-\frac{6T}{T-N-5}\\
&\Big[T+N-3+2T\Big(3\psi^2+\frac{N-1}{T}\Big)\Big]+\frac{2T}{(T-N)(T-N-3)}\Big[2(N+1)+T(T-N-3)\\
&+2T(T-N)\Big(3\psi^2+2\frac{N-1}{T}\Big)\Big]\geq 0. \qquad \text{(IA42)}
\end{aligned}
$$

Again, one can check that inequality (IA42) holds when $\psi^2=0$ under Assumption 1. Therefore, we prove as usual that the derivative of the left-hand side of (IA42) with respect to ψ^2 is positive. That is,

$$
\begin{aligned}
&\frac{N^3+2N^2T-6N^2-7NT^2+40NT-53N+4T^3-34T^2+88T-70}{(T-2)(T-N-2)(T-N-5)(T-N-7)}\\
&-\frac{6}{T-N-5}+\frac{2}{T-N-3}\geq 0. \qquad \text{(IA43)}
\end{aligned}
$$

This last inequality holds under Assumption 1, which concludes the demonstration of inequality (IA37) for $\kappa \geq \kappa_E^\star$.

Proof of Proposition 5

By definition of ψ^2 in (7), it can be written as

$$\psi^2 = \mu^\top \boldsymbol{\Sigma}^{-1}\mu - \frac{(e^\top \boldsymbol{\Sigma}^{-1}\mu)^2}{e^\top \boldsymbol{\Sigma}^{-1} e}. \tag{IA44}$$

Using the eigenvalue decomposition $\boldsymbol{\Sigma}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^\top$ and exploiting Assumption 3, the three quantities appearing in (IA44) simplify to

$$\begin{aligned}
\mu^\top \boldsymbol{\Sigma}^{-1}\mu &= \mu^\top \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^\top \mu = \sum_{i=1}^{N} \frac{(v_i^\top \mu)^2}{d_i}, \\
e^\top \boldsymbol{\Sigma}^{-1}\mu &= \sqrt{N} v_1^\top \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^\top \mu = \sqrt{N}\frac{v_1^\top \mu}{d_1}, \\
e^\top \boldsymbol{\Sigma}^{-1} e &= N v_1^\top \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^\top v_1 = \frac{N}{d_1}.
\end{aligned}$$

Using these expressions, it is straightforward to check that (IA44) is equal to $\sum_{i>1}^{N} SR_{PC_i}^2$, which proves the proposition.

Proof of Proposition 6

For conciseness, we denote $U(\kappa) = U(\hat{w}(\kappa))$. The covariance between $\hat{\psi}^2 = \hat{\mu}^\top \widehat{\boldsymbol{\Sigma}}^{-1}\hat{\mu} - (\hat{\mu}_g/\hat{\sigma}_g)^2$ and $(U(\kappa) - \mathbb{E}[U(\kappa)])^2$ is

$$\mathbb{C}\mathrm{ov}\Big[\hat{\psi}^2, (U(\kappa) - \mathbb{E}[U(\kappa)])^2\Big] = \mathbb{E}\Big[\hat{\psi}^2 (U(\kappa) - \mathbb{E}[U(\kappa)])^2\Big] - \mathbb{E}[\hat{\psi}^2]\mathbb{V}[U(\kappa)]. \tag{IA45}$$

In Equation (IA45), the OOSU variance $\mathbb{V}[U(\kappa)]$ is given in Proposition 3. Moreover, from Kan and Zhou (2007), we have

$$\mathbb{E}[\hat{\psi}^2] = \frac{\psi^2 T + N - 1}{T - N - 1}. \tag{IA46}$$

The last expectation, $\mathbb{E}\Big[\hat{\psi}^2(U(\kappa)-\mathbb{E}[U(\kappa)])^2\Big]$, decomposes as

$$\mathbb{E}\Big[\hat{\psi}^2(U(\kappa)-\mathbb{E}[U(\kappa)])^2\Big]=\mathbb{E}\Big[\hat{\psi}^2U(\kappa)^2\Big]-2\mathbb{E}[U(\kappa)]\mathbb{E}\Big[\hat{\psi}^2U(\kappa)\Big]+\mathbb{E}[U(\kappa)]^2\mathbb{E}[\hat{\psi}^2]. \tag{IA47}$$

In Equation (IA47), the OOSU mean $\mathbb{E}[U(\kappa)]$ is given in Proposition 2. Therefore, it remains to evaluate two expectations: $\mathbb{E}\Big[\hat{\psi}^2U(\kappa)\Big]$ and $\mathbb{E}\Big[\hat{\psi}^2U(\kappa)^2\Big]$. Using the stochastic representation for $\hat{\psi}^2$ and $U(\kappa)$ available in Kan et al. (2021b, Proposition 1), we find the following analytical expressions for the two expectations:

$$\begin{aligned}
\mathbb{E}\Big[\hat{\psi}^2U(\kappa)\Big]&=\frac{\psi^2T+N-1}{T-N-1}\left(\mu_g-\frac{\gamma\sigma_g^2(T-2)(T-N-2)}{2(T-N)(T-N-3)}\right)\\
&\quad+\frac{\kappa\psi^2T(\psi^2T+N+1)}{\gamma(T-N-1)(T-N-3)}\\
&\quad-\frac{\kappa^2T(T-2)\Big(\psi^4T^2+2\psi^2T(N+1)+(N+1)(N-1)\Big)}{2\gamma(T-N)(T-N-1)(T-N-3)(T-N-5)},
\end{aligned}\tag{IA48}$$

$$\begin{aligned}
\mathbb{E}\Big[\hat{\psi}^2U(\kappa)^2\Big]&=\frac{\mu_g^2(\psi^2T+N-1)}{T-N-1}+\frac{\sigma_g^2\psi^2\Big(\psi^2T(T-N)+N(T-N-1)+4-T\Big)}{(T-N)(T-N-1)(T-N-3)}\\
&\quad-\frac{\gamma\mu_g\sigma_g^2(T-2)(T-N-2)(\psi^2T+N-1)}{(T-N)(T-N-1)(T-N-3)}\\
&\quad+\frac{\gamma^2\sigma_g^4(T-2)(T-4)(\psi^2T+N-1)((T-N-1)(T-N-5)+3)}{4(T-N)(T-N-1)(T-N-2)(T-N-3)(T-N-5)}\\
&\quad+\frac{\kappa\psi^2T(\psi^2T+N+1)\Big(\frac{2\mu_g}{\gamma}(T-N)(T-N-5)-\sigma_g^2(T-2)(T-N-2)\Big)}{(T-N)(T-N-1)(T-N-3)(T-N-5)}\\
&\quad+\frac{\kappa^2\psi^2T\Big(\psi^4T^2(T-N)+\psi^2T((T-N)(N+4)+N-2)+(N-1)(T-2)\Big)}{\gamma^2(T-N)(T-N-1)(T-N-3)(T-N-5)}\\
&\quad-\frac{\kappa^2\mu_gT(T-2)\Big(\psi^4T^2+2\psi^2(N+1)+(N+1)(N-1)\Big)}{\gamma(T-N)(T-N-1)(T-N-3)(T-N-5)}\\
&\quad+\frac{\kappa^2\sigma_g^2T(T-2)(T-4)(T-N-4)\Big(\psi^4T^2+2\psi^2T(N+1)+(N+1)(N-1)\Big)}{2(T-N)(T-N-1)(T-N-2)(T-N-3)(T-N-5)(T-N-7)}\\
&\quad-\frac{\kappa^3\psi^2T^2(T-2)\Big(\psi^4T^2+2\psi^2T(N+3)+(N+3)(N+1)\Big)}{\gamma^2(T-N)(T-N-1)(T-N-3)(T-N-5)(T-N-7)}\\
&\quad+\frac{\kappa^4T^2(T-2)(T-4)/(T-N-9)}{4\gamma^2(T-N)(T-N-1)(T-N-2)(T-N-3)(T-N-5)(T-N-7)}\\
&\quad\times\Big(\psi^6T^3+3\psi^4T^2(N+3)+3\psi^2T(N+3)(N+1)+(N+3)(N+1)(N-1)\Big),
\end{aligned}\tag{IA49}$$

where $\mathbb{E}\left[\hat{\psi}^2 U(\kappa)^2\right]$ exists under the assumption that $T > N+9$.

To prove that the covariance is always positive, note first that it is independent of μ_g because $U(\kappa) - \mathbb{E}[U(\kappa)]$ is independent of μ_g. Therefore, without loss of generality, we set $\mu_g = 0$. We begin by showing that the covariance increases with σ_g^2, and thus, that it suffices to show that the covariance is positive for $\sigma_g^2 = 0$. To show that the covariance increases with σ_g^2, we take the derivative with respect to σ_g^2 and observe that it is positive when $\sigma_g^2 = 0$, and moreover the derivative itself increases with σ_g^2. Indeed, we have

$$\begin{aligned}
&\frac{\partial^2}{\partial\sigma_g^2\partial\sigma_g^2}\mathbb{C}\mathrm{ov}\left[\hat{\psi}^2, (U(\kappa) - \mathbb{E}[U(\kappa)])^2\right] = \\
&\frac{\gamma^2(T-2)(\psi^2 T + N - 1)}{2(T-N)(T-N-1)^3(T-N-2)(T-N-3)(T-N-5)} \\
&\times\Big((T-4)(T-N-1)^2((T-N-1)(T-N-5)+3) - 2(T-2)(T-N-2)(T-N-5)\Big),
\end{aligned} \tag{IA50}$$

which is positive under the assumption that $T > N+9$. Therefore, what remains is to show that the covariance is positive for $\mu_g = 0$ and $\sigma_g^2 = 0$. We proceed in a similar way to the proof of Proposition 4 by showing that the covariance is positive for $\kappa = 0$, taking derivatives with respect to κ, and showing each time that these derivatives are positive for $\kappa = 0$. At the end of this process, what remains to show is that the following quantity is positive:

$$\begin{aligned}
&\frac{(T-4)\Big(\psi^6 T^3 + 3\psi^4 T^2(N+3) + 3\psi^2 T(N+3)(N+1) + (N+3)(N+1)(N-1)\Big)}{(T-N-2)(T-N-5)(T-N-7)(T-N-9)} \\
&- \frac{2(T-2)(\psi^2 T + N - 1)\Big(\psi^4 T^2 + 2\psi^2 T(N+1) + (N+1)(N-1)\Big)}{(T-N)(T-N-1)(T-N-3)(T-N-5)} \\
&+ \frac{(T-2)(\psi^2 T + N - 1)^3}{(T-N)(T-N-1)^2(T-N-3)} \\
&- \frac{2C(T,N,\psi^2)(\psi^2 T + N - 1)}{(T-N)(T-N-1)^2(T-N-2)(T-N-3)(T-N-5)(T-N-7)} \geq 0,
\end{aligned} \tag{IA51}$$

where $C(T,N,\psi^2)$ is defined in Proposition 3. To show that inequality (IA51) holds, observe that it holds for $\psi^2 = 0$, and we can proceed as before by taking iterative derivatives with respect to ψ^2 and showing that they are positive for $\psi^2 = 0$. At the end of this process, what remains to show is the following inequality:

$$\begin{aligned}
&\frac{T-4}{(T-N-2)(T-N-5)(T-N-7)(T-N-9)} \\
&- \frac{2(T-2)}{(T-N)(T-N-1)(T-N-3)(T-N-5)} \\
&+ \frac{T-2}{(T-N)(T-N-1)^2(T-N-3)} \\
&- \frac{2(N^3+2N^2T-6N^2-7NT^2+40NT-53N+4T^3-34T^2+88T-70)}{(T-N)(T-N-1)^2(T-N-2)(T-N-3)(T-N-5)(T-N-7)} \geq 0. \quad \text{(IA52)}
\end{aligned}$$

This inequality holds under the assumption $T > N + 9$, which concludes the proof.

Proof of Proposition 7

Part 1. The proof is direct because, as shown in Proposition 3, the OOSU variance $\mathbb{V}[U(\hat{w}^\star(\kappa))] \to 0$ as $T \to \infty$ and, thus, the shrinkage intensity $\kappa_R^\star$ corresponds to $\kappa_E^\star$ as $T \to \infty$, and as shown in Proposition 2 this $\kappa_E^\star$ is asymptotically optimal.

Part 2. First, $\kappa_R^\star \geq \kappa_V^\star$ because $\kappa_V^\star$ minimizes OOSU variance by definition and the OOSU mean in (16) is increasing in κ for $\kappa \leq \kappa_E^\star$. Since $\kappa_V^\star \leq \kappa_E^\star$ as we will prove next, this means that $\kappa_V^\star$ has a larger OOSU mean and smaller OOSU variance than any $\kappa \leq \kappa_V^\star$. Therefore, $\kappa_R^\star$ maximizing the mean-risk OOSU metric in (27) is necessarily larger than $\kappa_V^\star$.

Second, we prove the inequality $\kappa_R^\star \leq \kappa_E^\star$, which also implies $\kappa_V^\star \leq \kappa_E^\star$. To prove this inequality, note from part 2 of Proposition 4 that OOSU variance increases with κ if $\kappa \geq \kappa_E^\star$. Moreover, OOSU mean in (16) is decreasing in κ for $\kappa \geq \kappa_E^\star$. As a result, any $\kappa \geq \kappa_E^\star$ delivers a smaller mean-risk OOSU than $\kappa_E^\star$ and thus $\kappa_R^\star$ is necessarily smaller than $\kappa_E^\star$.

Proof of Proposition 8

Parts 1 and 2. The proof is direct because, on the one hand, the OOSU mean in (16) increases with T and μ_g and decreases with N, σ_g^2, and κ if $\kappa \geq \kappa_E^\star$. On the other hand, we show in Proposition 4 that OOSU standard deviation decreases with T and κ if $\kappa \geq \kappa_E^\star$, increases with N and σ_g^2, and also is independent of μ_g.

Part 3. The derivative of the robustness measure with respect to λ is $\frac{\partial}{\partial \lambda} R(\hat{w}^\star(\kappa)) =$

$-\sqrt{\mathbb{V}[U(\hat{w}^\star(\kappa))]}$ and, as shown in Proposition 4, the OOSU variance $\mathbb{V}[U(\hat{w}^\star(\kappa))]$ increases with ψ^2, which proves the result.

Proof of Proposition IA.1

Denote $\hat{\mu}_g = \hat{w}_g^\top \hat{\mu}$ and $\hat{\sigma}_g^2 = \hat{w}_g^\top \widehat{\mathbf{\Sigma}} \hat{w}_g$. Then, the coefficients A and B in (IA2) correspond to $A = 1/\hat{\sigma}_g^2$ and $B = \hat{\mu}_g/\hat{\sigma}_g^2$. Denote also $f(\varepsilon) = 1 + \varepsilon/(\gamma\sigma_P^\star)$. Then, the ambiguity-averse portfolio can be rewritten as

$$\hat{w}^\star(\varepsilon) = \frac{1}{\gamma f(\varepsilon)} \widehat{\mathbf{\Sigma}}^{-1}\Big(\hat{\mu} - \hat{\mu}_g e + \gamma f(\varepsilon)\hat{\sigma}_g^2 e\Big) = \hat{w}_g + \frac{1}{\gamma f(\varepsilon)} \widehat{\mathbf{\Sigma}}^{-1}(\hat{\mu} - \hat{\mu}_g e).$$

The result follows by noticing that $\widehat{\mathbf{\Sigma}}^{-1}(\hat{\mu} - \hat{\mu}_g e) = \widehat{\mathbf{B}}\hat{\mu} = \hat{w}_z$, which is the estimated zero-cost portfolio, and therefore $\hat{w}^\star(\varepsilon)$ corresponds to the shrinkage portfolio $\hat{w}^\star(\kappa)$ in (11) if $\kappa = 1/f(\varepsilon) = (1 + \frac{\varepsilon}{\gamma\sigma_P^\star})^{-1}$.

Finally, $\sigma_P^\star$ is monotonically decreasing in ε because Garlappi et al. (2007) show that a higher ε implies a higher exposure to the SGMV portfolio and, thus, a smaller portfolio-return volatility. Consequently, the ratio $\varepsilon/\sigma_P^\star$ is monotonically increasing in ε.

Proof of Proposition IA.2

To prove the results in this proposition, we use Okhrin and Schmid (2006, Theorem 1) to find that, under Assumptions 1 and 2, the mean and covariance matrix of the shrinkage portfolio $\hat{w}^\star(\kappa)$ in (11) are

$$\mathbb{E}[\hat{w}^\star(\kappa)] = w_g + \frac{\kappa}{\gamma}\frac{T}{T-N-1}\mathbf{B}\mu, \tag{IA53}$$

$$\begin{aligned}\mathbb{V}[\hat{w}^\star(\kappa)] =& \left(\frac{\sigma_g^2}{T-N-1} + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)+T^2\psi^2}{(T-N)(T-N-1)(T-N-3)}\right)\mathbf{B} \\ &+ \frac{\kappa^2}{\gamma^2}\frac{T^2(T-N+1)}{(T-N)(T-N-1)^2(T-N-3)}\mathbf{B}\mu\mu^\top\mathbf{B}.\end{aligned} \tag{IA54}$$

Moreover, we use the following useful properties: $\mathbf{B}e = \mathbf{0}$, $\mathbf{B\Sigma B} = \mathbf{B}$, $\mu^\top \mathbf{B\Sigma} w_{ew} = \mu_{ew} - \mu_g$, and $w_{ew}^\top \mathbf{\Sigma B \Sigma} w_{ew} = \sigma_{ew}^2 - \sigma_g^2$.

Part 1. The OOSU mean of the shrinkage portfolio $\hat{w}^\star(\pi,\kappa)$ in (IA10) is

$$\mathbb{E}[U(\hat{w}^\star(\pi,\kappa))] = (1-\pi)\mu_{ew} + \pi\mathbb{E}\Big[\hat{w}^\star(\kappa)^\top\mu\Big] - \frac{\gamma}{2}\bigg((1-\pi)^2\sigma_{ew}^2 + \pi^2\mathbb{E}\Big[\hat{w}^\star(\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big] + 2\pi(1-\pi)\mathbb{E}\Big[w_{ew}^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big]\bigg). \tag{IA55}$$

From Kan et al. (2021b, Lemma 1), we have

$$\mathbb{E}\Big[\hat{w}^\star(\kappa)^\top\mu\Big] = \mu_g + \frac{\kappa}{\gamma}\frac{T}{T-N-1}\psi^2, \tag{IA56}$$

$$\mathbb{E}\Big[\hat{w}^\star(\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big] = \frac{T-2}{T-N-1}\sigma_g^2 + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)(N-1)+T^2(T-2)\psi^2}{(T-N)(T-N-1)(T-N-3)}. \tag{IA57}$$

Moreover, using Equation (IA53), we find that

$$\mathbb{E}\Big[w_{ew}^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big] = \sigma_g^2 + \frac{\kappa}{\gamma}\frac{T}{T-N-1}(\mu_{ew}-\mu_g). \tag{IA58}$$

Plugging (IA56)–(IA58) into (IA55), we find that the OOSU mean is given by Equation (IA12).

Part 2. We derive the expressions for the three components of the OOSU variance in (IA13).

First, the variance of out-of-sample mean return is

$$\mathbb{V}\Big[\hat{w}^\star(\pi,\kappa)^\top\mu\Big] = \pi^2\mathbb{V}\Big[\hat{w}^\star(\kappa)^\top\mu\Big],$$

where $\mathbb{V}\Big[\hat{w}^\star(\kappa)^\top\mu\Big]$ is given by (IA21), which results in Equation (IA14).

Second, the variance of out-of-sample return variance is

$$\mathbb{V}\Big[\hat{w}^\star(\pi,\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\pi,\kappa)\Big] = \pi^4\mathbb{V}\Big[\hat{w}^\star(\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big] + 4\pi^2(1-\pi)^2 w_{ew}^\top\mathbf{\Sigma}\mathbb{V}[\hat{w}^\star(\kappa)]\mathbf{\Sigma}w_{ew} + 4\pi^3(1-\pi)\mathbb{C}\text{ov}\Big[\hat{w}^\star(\kappa)^\top\mathbf{\Sigma}w_{ew}, \hat{w}^\star(\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big]. \tag{IA59}$$

The first term, $\mathbb{V}\Big[\hat{w}^\star(\kappa)^\top\mathbf{\Sigma}\hat{w}^\star(\kappa)\Big]$, is given by (IA22). Using Equation (IA54), the second term is

$$w_{ew}^{\top}\mathbf{\Sigma}\mathbb{V}[\hat{w}^{\star}(\kappa)]\mathbf{\Sigma}w_{ew} = (\sigma_{ew}^2 - \sigma_g^2)\left(\frac{\sigma_g^2}{T-N-1} + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)+T^2\psi^2}{(T-N)(T-N-1)(T-N-3)}\right) + \frac{\kappa^2}{\gamma^2}\frac{T^2(T-N+1)}{(T-N)(T-N-1)^2(T-N-3)}(\mu_{ew}-\mu_g)^2.$$

The third term is similar to (IA23) and is

$$\mathbb{C}\mathrm{ov}\Big[\hat{w}^{\star}(\kappa)^{\top}\mathbf{\Sigma}w_{ew}, \hat{w}^{\star}(\kappa)^{\top}\mathbf{\Sigma}\hat{w}^{\star}(\kappa)\Big] = \frac{2\kappa}{\gamma}(\mu_{ew}-\mu_g)\left(\sigma_g^2\frac{T(T-2)}{(T-N-1)^2(T-N-3)} + \frac{\kappa^2}{\gamma^2}\frac{T^2(T-2)(T+N-3+2T\psi^2)}{(T-N)(T-N-1)^2(T-N-3)(T-N-5)}\right). \quad \text{(IA60)}$$

Putting these three terms together into (IA59) gives Equation (IA15).

Third, the covariance between out-of-sample mean return and variance is

$$\mathbb{C}\mathrm{ov}\Big[\hat{w}^{\star}(\pi,\kappa)^{\top}\mu, \hat{w}^{\star}(\pi,\kappa)^{\top}\mathbf{\Sigma}\hat{w}^{\star}(\pi,\kappa)\Big] = \pi^3\mathbb{C}\mathrm{ov}\Big[\hat{w}^{\star}(\kappa)^{\top}\mu, \hat{w}^{\star}(\kappa)^{\top}\mathbf{\Sigma}\hat{w}^{\star}(\kappa)\Big] + 2\pi^2(1-\pi)\mathbb{C}\mathrm{ov}\Big[\hat{w}^{\star}(\kappa)^{\top}\mu, \hat{w}^{\star}(\kappa)^{\top}\mathbf{\Sigma}w_{ew}\Big]. \quad \text{(IA61)}$$

The first term, $\mathbb{C}\mathrm{ov}\Big[\hat{w}^{\star}(\kappa)^{\top}\mu, \hat{w}^{\star}(\kappa)^{\top}\mathbf{\Sigma}\hat{w}^{\star}(\kappa)\Big]$, is given by (IA23). Using Equation (IA54), the second term is

$$\mathbb{C}\mathrm{ov}\Big[\hat{w}^{\star}(\kappa)^{\top}\mu, \hat{w}^{\star}(\kappa)^{\top}\mathbf{\Sigma}w_{ew}\Big] = \mu^{\top}\mathbb{V}[\hat{w}^{\star}(\kappa)\mathbf{\Sigma}w_{ew} = (\mu_{ew}-\mu_g)\left(\frac{\sigma_g^2}{T-N-1} + \frac{\kappa^2}{\gamma^2}\frac{T(T-2)(T-N-1)+2T^2(T-N)\psi^2}{(T-N)(T-N-1)^2(T-N-3)}\right).$$

Putting these two terms together into (IA61) results in the final expression in Equation (IA16) and concludes the proof.

Proof of Proposition IA.3

To prove this proposition, we use Okhrin and Schmid (2006, Theorem 1), who show that if $T > N$, $N \geq 2$, and Assumption 2 holds, then the mean and covariance matrix of the SGMV portfolio $\hat{w}_g$ are

$$\mathbb{E}[\hat{w}_g] = w_g \quad \text{and} \quad \mathbb{V}[\hat{w}_g] = \frac{\sigma_g^2}{T-N-1}\mathbf{B}.$$

Using this result, the mean and covariance matrix of the shrinkage portfolio $\hat{w}(\pi) = \pi w_{ew} + (1-\pi)\hat{w}_g$ are

$$\begin{aligned}
\mathbb{E}[\hat{w}(\pi)] &= \pi\mu_{ew} + (1-\pi)\mu_g, \\
\mathbb{V}[\hat{w}(\pi)] &= (1-\pi)^2\frac{\sigma_g^2}{T-N-1}\mathbf{B}.
\end{aligned}$$

Therefore, the mean squared error $\mathbb{E}[(\hat{w}(\pi)^\top\mu - \mu_g)^2]$ is given by

$$\begin{aligned}
\mathbb{E}\Big[(\hat{w}(\pi)^\top\mu - \mu_g)^2\Big] &= \Big(\mathbb{E}\Big[\hat{w}(\pi)^\top\mu\Big] - \mu_g\Big)^2 + \mathbb{V}\Big[\hat{w}(\pi)^\top\mu\Big] \\
&= \pi^2(\mu_{ew} - \mu_g)^2 + (1-\pi)^2\frac{\sigma_g^2\psi^2}{T-N-1}.
\end{aligned}$$

Taking the derivative of $\mathbb{E}[(\hat{w}(\pi)^\top\mu - \mu_g)^2]$ with respect to π and setting it to zero yields the final expression for π in (IA18).

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