American-type basket option pricing: a simple two-dimensional partial differential equation[1]

Hamza Hanbali & Daniël Linders

University of Nebraska-Lincoln, USA
Snell Actuarial Science and Risk Management Seminar Series

October 31, 2019

[1]Based on Hanbali, H. & Linders, D. (2019) 'American-type basket option pricing: a simple two-dimensional partial differential equation', *Quantitative Finance*, 2019 Vol. 19, No. 10, 1689-1704.

Definition

- The stock basket:
  - the price of stock j at time t is denoted by $S_j(t)$.
  - the price of the **basket** at time t is denoted by $S(t)$:

$$S(t) = w_1 S_1(t) + \cdots + w_n S_n(t), \quad w_j \geq 0.$$

- Basket derivative:
  - start of the contract: $t = 0$;
  - pay-off function H and maturity T;
  - arbitrage-free price:

$$\text{time-}t \text{ price} = V(t, S_1, S_2, \ldots, S_n).$$

Examples

- European-type basket derivative:
  - H is a function of $S(T)$ only.
  - Example: basket call and put options.
- Path-dependent basket derivative:
  - H is a function of the process S between time 0 and time T.
  - Example: Barrier and Asian basket options.
- American-type basket derivative:
  - Exercising the options of the contract is possible at any time $t \leq T$.
  - Example: American-type Asian basket options.

- Our focus will be on **basket options**.
- Basket call option:
  - Strike K and maturity T;
  - Pay-off function:

$$\text{Pay-off} = \max\{S(T) - K, 0\} \overset{notation}{=} (S(T) - K)_+.$$

- Basket put option:
  - Strike K and maturity T;
  - Pay-off function:

$$\text{Pay-off} = \max\{K - S(T), 0\} \overset{notation}{=} (K - S(T))_+.$$

- Why do we need basket options?
  - Basket options can provide protection for an investment portfolio.
  - Basket options can be used to construct forward-looking market implied dependence measures.
  - Dispersion trading is the most popular strategy for trading correlation and involves basket options.

Basket options: references

- Multivariate Black & Scholes:
  - Milevsky & Posner (1998), Krekel, Kock, Korn & Man (2002), Brigo, Mercurio, Rapisarda, & Scotti (2004), Deelstra, Liinev & Vanmaele (2004), Borovkova, Permana & V.D. Weide (2007), Linders & Schoutens (2014).
- Non-Gaussian models:
  - Xu & Zheng (2014), Leccadito, Paletta & Tunaru (2016), Caldana, Fusai, Gnoatto & Graselli (2016), Bo & Wang (2017), Linders & Stassen (2016), Linders & Schoutens (2016).
- Model-free upper bounds:
  - Hobson, Laurence & Wang (2005), D'Aspremont & El Ghaoui (2006), Chen, Deelstra, Dhaene & Vanmaele (2008), Linders, Dhaene, Hounnon & Vanmaele (2012).

Multivariate Black & Scholes market

- The multivariate Black & Scholes model[2]:

$$\frac{\mathrm{d}S_i(t)}{S_i(t)} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}B_i(t), \text{ for } t > 0 \text{ and } i = 1, 2, \ldots, n,$$

  - μ_i is the drift of stock i,
  - σ_i is the volatility of stock i,
  - B_i is a standard Brownian motion.

- Correlation:

$$\mathbb{E}\left[\mathrm{d}B_i(t)\mathrm{d}B_j(t)\right] = \rho_{i,j}\mathrm{d}t.$$

[2] The usual assumptions apply: the risk-free rate r is constant and there exists a risk-neutral measure $\mathbb{Q}$.

Pricing methodologies

- Methodology 1: PDE approach
  - Solve the Partial Differential Equation combined with the appropriate final condition:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{i,j}w_iw_jS_iS_j\frac{\partial^2 V}{\partial S_i\partial S_j} + r\sum_{i=1}^{n}S_i\frac{\partial V}{\partial S_i} - rV = 0.$$

  - Problem:

Very hard to solve numerically!

- Methodology 2: Risk-neutral valuation
  - Determine the discounted risk-neutral expectation:

$$V(t, S_1, S_2, \ldots, S_n) = \mathrm{e}^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[H(S(T))\middle|\, \mathcal{F}_t\right].$$

  - Problem:

The distribution of $S(T)$ is unknown!

Risk-neutral valuation: pros and cons

- Pros of risk-neutral valuation:
  - Only the time-T distribution is needed.
  - Efficient in **high dimensions**.
- Cons of risk-neutral valuation:
  - Difficult to include **early exercise**.
  - Monte Carlo simulation is slow in low dimensions.

PDE approach: pros and cons

- Pros of the PDE approach:
  - **Early exercise** features can easily be incorporated using finite difference methods.
  - Strong path dependent derivatives can be priced. (e.g. Asian options)
- Cons of the PDE approach:
  - Difficult in **high dimensions**.
  - Dynamics of the stock prices are needed.

Pricing using the PDE approach

- Our focus:

  pricing **multivariate** derivatives using the **PDE** approach.

- Curse of dimensionality:
  - Problem: mixed derivative terms;
  - numerical solutions are hard to implement.
- Numerically solving the multidimensional PDE:
  - Remove the mixed terms by changing the coordinate system.
    - Company, Egorova, Jodar & Soleymani (2016), Riesinger & Wissman (2017).
  - Sparse grid method: combine smaller grid solutions
    - Leentvaar & Oosterlee (2008).

Aim of the paper

- **The comonotonic market:**
  - Tractable PDE for the basket derivative price;
  - We can derive an *exact solution* which is fast to determine;
  - Comonotonic *finite difference* scheme is efficient, even in high dimensions.
- **Approximate basket derivative pricing:**
  - *Approximate* a non-comonotonic market with an artificial comonotonic market;
  - Pricing in the artificial comonotonic market is *fast and efficient*;
  - Basket derivative prices in the artificial market are accurate approximations for the real basket derivative prices.

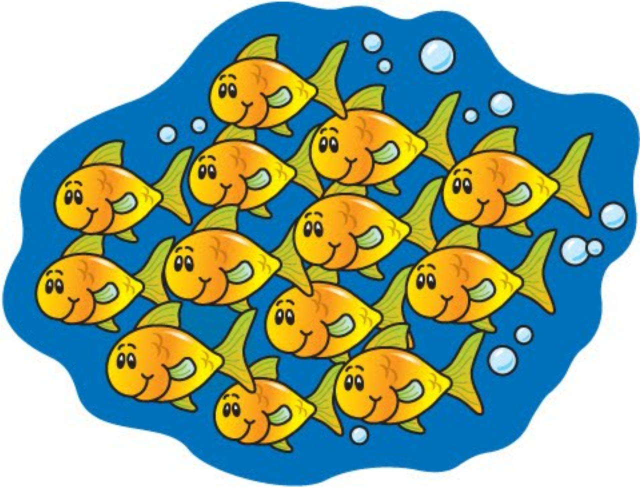
Introduction

- <u>The comonotonic market:</u>
  - stock prices at time t are denoted by $S_i^c(t)$:

$$\frac{\mathrm{d}S_i^c(t)}{S_i^c(t)} = \mu_i \mathrm{d}t + \sigma_i \mathrm{d}B(t), \text{ for } t > 0 \text{ and } i = 1, 2, \ldots, n.$$

  - B is a standard Brownian motion;
  - the marginal distributions: $S_i^c(t) \stackrel{d}{=} S_i(t)$.
- <u>Copula:</u>
  - comonotonic copula;
  - in a comonotonic market, all stocks are driven by the same single Brownian motion B.

Introduction



Notation

- The comonotonic basket:

$$S^c(t) = w_1 S_1^c(t) + \ldots + w_n S_n^c(t).$$

- Lemma:
  - The SDE of the comonotonic basket S^c is given by

$$dS^c(t) = \mu^c(t, B)dt + \sigma^c(t, B)dB(t),$$

  - where

$$\mu^c(t, B) = \sum_{i=1}^{n} \mu_i w_i S_i^c(t, B)$$

  - and

$$\sigma^c(t, B) = \sum_{i=1}^{n} \sigma_i w_i S_i^c(t, B).$$

3 – The comonotonic market

The shifted Brownian motion process

- The following statements are equivalent[3]:
  - The comonotonic market is arbitrage-free.
  - There exists a λ satisfying:

$$\lambda = \frac{\mu_i - r}{\sigma_i}, \text{ for all } i = 1, 2, \ldots, n.$$

- Interpretation:
  - the random source of each stock is the same;
  - one unit volatility is essentially the same for each stock;
  - therefore, each stock has the same market price of risk.

[3] see Dhaene, Kukush & Linders (2013)

The shifted Brownian motion

- The shifted Brownian motion process B_λ:

$$\mathrm{d}B_\lambda(t) = \mathrm{d}B(t) + \lambda \mathrm{d}t.$$

- The comonotonic stock prices:

$$S_i^c(t) = S_i(0)\mathrm{e}^{\left(r-\frac{1}{2}\sigma_i^2\right)t+\sigma_i B_\lambda(t)}, \text{ for } i = 1, 2, \ldots, n.$$

- The dynamics of the comonotonic basket:

$$\mathrm{d}S^c(t) = rS^c(t)\mathrm{d}t + \sigma^c(t, B)\mathrm{d}B_\lambda(t).$$

A risk-free portfolio

- The following statements are equivalent:
  - observing the realization $B_\lambda(t)$;
  - observing the realization $S_i^c(t)$;
  - observing the realization $S^c(t)$.
- Price of the derivative at maturity time T:
  - $V^c\left(t, S^c(t)\right)$
- Price of the derivative in function of B_λ:

$$V_\lambda^c\left(t, B_\lambda\right) = V^c\left(t, \sum_{i=1}^{n} w_i S_i(0) \mathrm{e}^{\left(r - \frac{1}{2}\sigma_i^2\right)t + \sigma_i B_\lambda}\right).$$

A risk-free portfolio

- Portfolio:
  - long one basket derivative V_λ^c;
  - short Δ units of the basket S^c.
- Time-t value of the portfolio:

$$\Pi(t) = V_\lambda^c(t) - \Delta S^c(t).$$

- Change in portfolio value:

$$\mathrm{d}\Pi = \mathrm{d}V_\lambda^c - \Delta \mathrm{d}S^c.$$

  - $\mathrm{d}S^c$: see previous lemma.
  - $\mathrm{d}V_\lambda^c$: use Ito's lemma.
- The portfolio is risk-free if:

$$\Delta = \frac{1}{\sigma^c}\frac{\partial V_\lambda^c}{\partial B_\lambda},$$

The PDE for the comonotonic basket derivative price

Theorem

The PDE for the derivative price V_λ^c:

$$\frac{\partial V_\lambda^c}{\partial t} + \frac{1}{2}\frac{\partial^2 V_\lambda^c}{\partial B_\lambda^2} - rV_\lambda^c = 0, \tag{1}$$

where the final condition is given by

$$V_\lambda^c\left(T, B_\lambda\right) = H\left(\sum_{i=1}^{n} w_i S_i(0) e^{\left(r-\frac{1}{2}\sigma_i^2\right)T + \sigma_i B_\lambda}\right). \tag{2}$$

Another PDE for the comonotonic basket derivative price

- The PDE in terms of the comonotonic basket:

$$\frac{\partial V^c}{\partial t'} + \frac{1}{2}\left(\sigma^c(t,B)\right)^2 \frac{\partial^2 V^c}{\partial \left(S^c\right)^2} + rS^c \frac{\partial V^c}{\partial S^c} - rV^c = 0.$$

- The final condition:

$$V^c(T, S^c) = H(S^c).$$

- Remarks:
  - Similar to the Black & Scholes PDE for one-dimensional derivative pricing.
  - Time-dependent volatility slows down the calculations.

The solution of the comonotonic PDE

Theorem

Closed-form solution for $V_\lambda^c(t, B_\lambda)$*:*

$$V_\lambda^c(t, B_\lambda) = e^{-r(T-t)} \int_{-\infty}^{+\infty} H\left(\sum_{i=1}^{n} w_i S_i^c(t) e^{\left(r-\frac{1}{2}\sigma_i^2\right)(T-t)+\sigma_i y}\right) \phi_{T-t}(y) dy, \quad (3)$$

where $S_i^c(t)$ *is given by*

$$S_i^c(t) = S_i(0) e^{\left(r-\frac{1}{2}\sigma_i^2\right)t+\sigma_i B_\lambda(t)}, \text{ for } i = 1, 2, \ldots, n, \quad (4)$$

and ϕ_{T-t} *is the density of a normal distribution with mean 0 and variance* $T-t$*:*

$$\phi_{T-t}(y) = \frac{e^{-\frac{y^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}.$$

The solution of the comonotonic PDE

- Weighted pay-off:
  - future realization of the increment of the process B_λ in $[t, T] = y$;
  - pay-off: $H\left(\sum_{i=1}^{n} w_i S_i^c(t) \mathrm{e}^{\left(r-\frac{1}{2}\sigma_i^2\right)(T-t)+\sigma_i y}\right)$;
  - Gaussian density in y:

$$g(y) = \frac{1}{\sqrt{2\pi(T-t)}} \mathrm{e}^{-\frac{y^2}{2(T-t)}}.$$

- Conclusion:
  - We determine the price $V_\lambda^c(t, B_\lambda)$ by integrating over all future states of the process B_λ;
  - weighted by the corresponding Gaussian probabilities.

The distribution of the comonotonic basket

Theorem

The price $V^c(t,S^c)$ is given by

$$V^c(t,S^c) = e^{-r(T-t)}\int_0^{+\infty} f_{S^c}(S';T,t)H(S')dS',$$

where

$$f_{S^c}(S';T,t) = \frac{1}{\sqrt{2\pi(T-t)}}\frac{e^{-\frac{\left(\Phi^{-1}\left(F_{S^c}(S';T,t)\right)\right)^2}{2}}}{\sum_{i=1}^n \sigma_i w_i S_i^c(t) e^{\left(r-\frac{1}{2}\sigma_i^2\right)(T-t)+\sigma_i\sqrt{T-t}\Phi^{-1}\left(F_{S^c}(S';T,t)\right)}},$$

is the time-t risk-neutral density of the comonotonic basket $S^c(T)$ and $F_{S^c}(S';T,t)$ is the solution of:

$$\sum_{i=1}^n w_i S_i^c(t) e^{\left(r-\frac{1}{2}\sigma_i^2\right)(T-t)+\sigma_i\sqrt{T-t}\Phi^{-1}\left(F_{S^c}(S';T,t)\right)} = S'.$$

Pricing formulas

- The price $V^c(t, S^c)$:
  - is an integral over all future realizations of the comonotonic basket $S^c(T)$;
  - weighted using the risk-neutral density;
  - is the discounted risk-neutral expectation.
- The price $V^c_\lambda(t, B_\lambda)$:
  - is an integral over all future realizations of the risk factor $B_\lambda(T)$;
  - weighted using a Gaussian density;
  - is the solution of a partial differential equation.

- A price V_λ^c can be obtained by numerically solving the PDE:

$$\frac{\partial V_\lambda^c}{\partial t} + \frac{1}{2}\frac{\partial^2 V_\lambda^c}{\partial B_\lambda^2} - rV_\lambda^c = 0 \tag{5}$$

- Discretisation:
  - Time grid:
    $$t_k = T - k\delta t, \text{ for } k = 0, 1, \ldots, L.$$
  - Brownian motion grid:
    $$b_j = (j - I)\delta B, \; j = 0, 1, \ldots, J.$$

- We determine basket derivative prices on the grid points:

$$V_j^k \equiv V_\lambda^c(t_k, b_j).$$

- Basket derivative prices at the maturity end points:

$$V_j^0 = H\left(\sum_{i=1}^{n} w_i S_i(0) \mathrm{e}^{\left(r-\frac{1}{2}\sigma_i^2\right)T+\sigma_i b_j}\right). \tag{6}$$

- Backwards explicit scheme:

$$V_j^{k+1} = \frac{1}{2}\frac{\delta t}{\delta B^2}\ V_{j-1}^k + \left(1 - r\delta t - \frac{\delta t}{\delta B^2}\right) V_j^k + \frac{1}{2}\frac{\delta t}{\delta B^2}\ V_{j+1}^k + \mathcal{O}(\delta t; \delta B^2),$$

for $j = 1, ..., J-1$ and $k = 1, ..., L$.

Remarks

- This scheme is
  - easy to implement;
  - converges to the real solution.
- Stability criterium:

$$\delta t \leq \frac{2\delta B^2}{r\delta B^2 + 2}.$$

- Increasing the accuracy:
  - Decrease the step size δB;
  - Increase the number of points J in the grid for the Brownian motion.

comonotonic American options

- American-type option:
  - The option can be exercised prior to maturity.
- Finite difference scheme for pricing American-type derivatives:
  - Time grid: $0 = t_L < t_{L-1} < \ldots < t_1 < t_0 = T$.
  - Assume: All prices are determined for $t > t_k$.
  - Goal: Determine the prices for t_{k+1}.
- In the interval $[t_{k+1}, t_k]$ there are two possibilities:
  - Possibility 1: The derivative is not exercised at time t_{k+1}.
  - Possibility 2: The derivative is exercised at time t_{k+1}.

comonotonic American options

- Possibility 1: The derivative is not exercises at time t_{k+1}.
  - The derivative behaves as a European-type derivative in $[t_{k+1}, t_k]$.
  - The price $\tilde{V}_j^{k+1}$ satisfies:

$$\tilde{V}_j^{k+1} \approx \frac{1}{2}\frac{\delta t}{\delta B^2}\ V_{j-1}^k + \left(1 - r\delta t - \frac{\delta t}{\delta B^2}\right) V_j^k + \frac{1}{2}\frac{\delta t}{\delta B^2}\ V_{j+1}^k.$$

- Possibility 2: The derivative is exercised at time t_{k+1}.
  - We receive the payoff:

$$H\left(S_j^{k+1}\right).$$

- American-type derivative price in the node $(j, k+1)$:

$$V_j^{k+1} = \max\left\{\tilde{V}_j^{k+1}, H\left(S_j^{k+1}\right)\right\}.$$

Approximating the financial market

- In a comonotonic market:
  - one risk-factor drives all stocks, the basket and the derivative prices;
  - basket derivative pricing is fast and efficient:
    - closed form expressions;
    - efficient comonotonic finite difference scheme.
- The real market situation:
  - the financial market is in general **not** comonotonic;
  - correlations are not equal to one;
  - assumption[4]: $\rho_{i,j} > 0$.
- Conclusion:

We didn't solve the complete problem!

[4] our methodology can be generalized such that this assumption can be relaxed.

Approximating the financial market

- Problem:
  - Determine the price $V(t)$ of a basket derivative;
  - where the components of the basket are **correlated**.
- We construct an **artificial** financial market:
  - the artificial market is required to be **comonotonic**:
    - closed-form expressions for basket derivative prices are available;
  - but: the stocks in the artificial market have an **adjusted volatility**:
    - the basket derivative price in the artificial market should be 'close' to the real price V.

Approximating the financial market

- The artificial financial market:

$$\frac{\mathrm{d}S_i^l(t)}{S_i^l(t)} = \mu_i \mathrm{d}t + \nu_i \sigma_i \mathrm{d}B(t), \text{ for } t > 0 \text{ and } i = 1, 2, \ldots, n,$$

where[5]

$$\nu_i = \frac{\sum_{j=1}^n w_j S_j(0) \rho_{i,j} \sigma_j}{\sqrt{\sum_{j=1}^n \sum_{k=1}^n w_j w_k S_j(0) S_k(0) \rho_{j,k} \sigma_j \sigma_k}}, \text{ for } i = 1, 2, \ldots, n.$$

- Remarks:
  - $0 < \nu_i \leq 1$.
  - Adjusted volatility: $r_i \sigma_i \leq \sigma_i$.

[5] other choices are possible: Deelstra, Liinev & Vanmaele (2004) and Hainaut & Deelstra (2014).

Approximating the financial market

- The marginal stock price process $\left\{S_i^l(t)|\ t \geq 0\right\}$
  - are following a Black & Scholes model;
  - **but:** with adjusted volatility parameter.
- Dependence = comonotonic copula
  - the artificial market is driven by the single Brownian motion B;
  - basket derivative pricing is fast and efficient:
    - closed-form solutions are available (single integration);
    - numerical methods are fast and accurate (comonotonic finite difference scheme).

Final approximation

- Approximate basket:

$$S^l(t) = w_1 S_1^l(t) + w_2 S_2^l(t) + \ldots + w_n S_n^l(t).$$

- Basket derivative price:

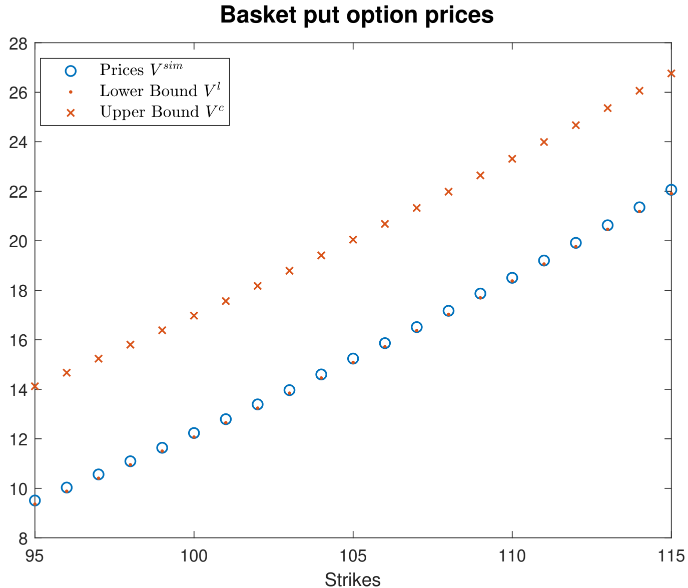
$$V^l(t, S).$$

Illustration

Table: Input parameters for the four-stock basket with Correlation $\rho = 0.3$.

	stock 1	stock 2	stock 3	stock 4
σ_i	0.5	0.2	0.8	0.9
$S_j(0)$	100	100	100	100
w_i	0.25	0.25	0.25	0.25

Illustration



Illustration

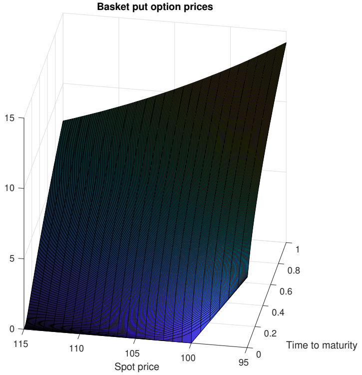


Figure: The approximation V^l for the basket option price V in function of the spot price $S(0)$ and the time-to-maturity T, together with the corresponding Δ of V^l.

Final approximation

- Final approximation for the European basket put option[6]:

$$\bar{V}(t, S_1, S_2, \ldots, S_n) = zV^l(t, S) + (1-z)V^c(t, S),$$

where

$$z = \frac{\mathrm{Var}_t\left[S^c(T)\right] - \mathrm{Var}_t\left[S(T)\right]}{\mathrm{Var}_t\left[S^c(T)\right] - \mathrm{Var}_t\left[S^l(T)\right]} \in [0, 1].$$

- The approximation $\bar{V}$ satisfies:

$$\int_0^\infty \bar{V} \mathrm{d}K = \int_0^\infty V \mathrm{d}K.$$

[6] Vyncke, Goovaerts & Dhaene (2004)

Illustration

- **Goal:** Determine the price of an American-type Basket put option.
  - Use Least-squares Monte Carlo Simulation to approximate the real price.
  - Use the approximation $\bar{V}$.

Table: American-type basket put option prices on a basket of 8 equally weighted stocks with initial prices 40 computed using finite difference (FD) method and the LSM.

Maturity	Strike	σ_1	ρ	FD prices	LSM prices	time FD	time LSM
2	35	0.3	0.3	4.035	3.974	37	729
			0.8	6.196	6.183	37	2374
		0.9	0.3	4.978	4.954	167	3028
			0.8	7.481	7.477	163	3454
	40	0.3	0.3	6.775	6.704	169	4023
			0.8	9.204	9.176	165	4148
		0.9	0.3	7.822	7.814	167	4055
			0.8	10.614	10.594	163	4245
	45	0.3	0.3	10.065	10.011	170	4990
			0.8	12.602	12.576	165	4862
		0.9	0.3	11.136	11.130	167	4911
			0.8	14.075	14.049	164	4836

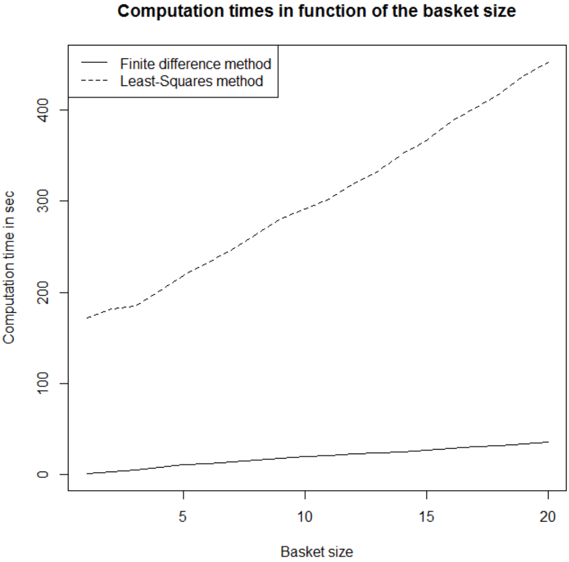

Computation times in function of the basket size
Finite difference method
Least-Squares method
Computation time in sec
0
100
200
300
400
5
10
15
20
Basket size

Table: American-type basket put option prices on a basket of 4 equally weighted stocks with initial prices 40 for high pairwise correlations.

Pairwise correlation	FD prices	LSM prices	Comonotonic LSM
0.95	7.427	7.433	–
0.96	7.452	7.444	–
0.97	7.476	7.476	–
0.98	7.500	7.511	–
0.99	7.524	8.670	–
0.992	7.529	10.250	–
0.994	7.534	10.234	–
0.996	7.538	14.702	–
0.998	7.543	17.212	–
1.000	7.548	259.274	7.550

Thank you for your attention!

Daniël Linders
dlinders@illinois.edu
www.daniellinders.com