American-type basket option pricing: a simple two-dimensional partial differential equation

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2. The financial market

3. The comonotonic market
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4. A comonotonic finite difference scheme

5. Approximate basket derivative pricing

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The stock basket:
- the price of stock $j$ at time $t$ is denoted by $S_j(t)$.
- the price of the basket at time $t$ is denoted by $S(t)$:
  $$S(t) = w_1S_1(t) + \cdots + w_nS_n(t), \quad w_j \geq 0.$$ 

Basket derivative:
- start of the contract: $t = 0$;
- pay-off function $H$ and maturity $T$;
- arbitrage-free price:
  $$\text{time-}t \ \text{price} = V(t, S_1, S_2, \ldots, S_n).$$
European-type basket derivative:
- $H$ is a function of $S(T)$ only.
- Example: basket call and put options.

Path-dependent basket derivative:
- $H$ is a function of the process $S$ between time 0 and time $T$.
- Example: Barrier and Asian basket options.

American-type basket derivative:
- Exercising the options of the contract is possible at any time $t \leq T$.
- Example: American-type Asian basket options.
Our focus will be on basket options.

Basket call option:
- Strike $K$ and maturity $T$;
- Pay-off function:

$$ \text{Pay-off} = \max \{ S(T) - K, 0 \} \overset{\text{notation}}{=} (S(T) - K)_+. $$

Basket put option:
- Strike $K$ and maturity $T$;
- Pay-off function:

$$ \text{Pay-off} = \max \{ K - S(T), 0 \} \overset{\text{notation}}{=} (K - S(T))_+. $$
Why do we need basket options?

- Basket options can provide protection for an investment portfolio.
- Basket options can be used to construct forward-looking market implied dependence measures.
- Dispersion trading is the most popular strategy for trading correlation and involves basket options.
Basket options: references

- **Multivariate Black & Scholes:**

- **Non-Gaussian models:**

- **Model-free upper bounds:**
The multivariate Black & Scholes model:

\[
\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dB_i(t), \text{ for } t > 0 \text{ and } i = 1, 2, \ldots, n,
\]

- \( \mu_i \) is the drift of stock \( i \),
- \( \sigma_i \) is the volatility of stock \( i \),
- \( B_i \) is a standard Brownian motion.

Correlation:

\[
E \left[ dB_i(t) dB_j(t) \right] = \rho_{i,j} dt.
\]

\[\text{2}\] The usual assumptions apply: the risk-free rate \( r \) is constant and there exists a risk-neutral measure \( Q \).
Methodology 1: PDE approach

- Solve the Partial Differential Equation combined with the appropriate final condition:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j \rho_{i,j} w_i w_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + r \sum_{i=1}^{n} S_i \frac{\partial V}{\partial S_i} - rV = 0.
\]

- Problem:

Very hard to solve numerically!

Methodology 2: Risk-neutral valuation

- Determine the discounted risk-neutral expectation:

\[
V(t, S_1, S_2, \ldots, S_n) = e^{-r(T-t)} \mathbb{E}_Q [H(S(T)) | \mathcal{F}_t].
\]

- Problem:

The distribution of $S(T)$ is unknown!
Pros of risk-neutral valuation:
- Only the time-$T$ distribution is needed.
- Efficient in high dimensions.

Cons of risk-neutral valuation:
- Difficult to include early exercise.
- Monte Carlo simulation is slow in low dimensions.
Pros of the PDE approach:

- Early exercise features can easily be incorporated using finite difference methods.
- Strong path dependent derivatives can be priced. (e.g. Asian options)

Cons of the PDE approach:

- Difficult in high dimensions.
- Dynamics of the stock prices are needed.
Our focus:

pricing multivariate derivatives using the PDE approach.

Curse of dimensionality:
- Problem: mixed derivative terms;
- numerical solutions are hard to implement.

Numerically solving the multidimensional PDE:
- Remove the mixed terms by changing the coordinate system.
- Sparse grid method: combine smaller grid solutions
Aim of the paper

- **The comonotonic market:**
  - Tractable PDE for the basket derivative price;
  - We can derive an *exact solution* which is fast to determine;
  - Comonotonic *finite difference* scheme is efficient, even in high dimensions.

- **Approximate basket derivative pricing:**
  - *Approximate* a non-comonotonic market with an artificial comonotonic market;
  - Pricing in the artificial comonotonic market is *fast and efficient*;
  - Basket derivative prices in the artificial market are accurate approximations for the real basket derivative prices.
The comonotonic market:

- stock prices at time $t$ are denoted by $S^c_i(t)$:

$$\frac{dS^c_i(t)}{S^c_i(t)} = \mu_i dt + \sigma_i dB(t), \text{ for } t > 0 \text{ and } i = 1, 2, \ldots, n.$$ 

- $B$ is a standard Brownian motion;

- the marginal distributions: $S^c_i(t) \overset{d}{=} S_i(t)$.

Copula:

- comonotonic copula;

- in a comonotonic market, all stocks are driven by the same single Brownian motion $B$. 

3 – The comonotonic market

Introduction
The comonotonic basket:

\[ S^c(t) = w_1 S^c_1(t) + \ldots + w_n S^c_n(t). \]

Lemma:
- The SDE of the comonotonic basket \( S^c \) is given by

\[ dS^c(t) = \mu^c(t, B) dt + \sigma^c(t, B) dB(t), \]

where

\[ \mu^c(t, B) = \sum_{i=1}^{n} \mu_i w_i S^c_i(t, B) \]

and

\[ \sigma^c(t, B) = \sum_{i=1}^{n} \sigma_i w_i S^c_i(t, B). \]
The following statements are equivalent:\(^3\):

- The comonotonic market is arbitrage-free.
- There exists a \( \lambda \) satisfying:
  \[
  \lambda = \frac{\mu_i - r}{\sigma_i}, \text{ for all } i = 1, 2, \ldots, n.
  \]

**Interpretation:**
- the random source of each stock is the same;
- one unit volatility is essentially the same for each stock;
- therefore, each stock has the same market price of risk.

\(^3\)see Dhaene, Kukush & Linders (2013)
The shifted Brownian motion process $B_\lambda$:

$$dB_\lambda(t) = dB(t) + \lambda dt.$$ 

The comonotonic stock prices:

$$S^c_i(t) = S_i(0)e^{(r - \frac{1}{2}\sigma^2_i)t + \sigma_iB_\lambda(t)}, \text{ for } i = 1, 2, \ldots, n.$$ 

The dynamics of the comonotonic basket:

$$dS^c(t) = rS^c(t)dt + \sigma^c(t, B)dB_\lambda(t).$$
A risk-free portfolio

- The following statements are equivalent:
  - observing the realization $B_\lambda(t)$;
  - observing the realization $S^c_i(t)$;
  - observing the realization $S^c(t)$.

- Price of the derivative at maturity time $T$:
  - $V^c(t, S^c(t))$

- Price of the derivative in function of $B_\lambda$:

$$V^c_\lambda(t, B_\lambda) = V^c \left( t, \sum_{i=1}^{n} w_i S_i(0) e^{\left(r - \frac{1}{2} \sigma^2_i\right)t + \sigma_i B_\lambda} \right) .$$
3 – The pde for the comonotonic basket derivative price 20/45

A risk-free portfolio

- Portfolio:
  - long one basket derivative $V^c_\lambda$;
  - short $\Delta$ units of the basket $S^c$.

- Time-$t$ value of the portfolio:
  \[ \Pi(t) = V^c_\lambda(t) - \Delta S^c(t). \]

- Change in portfolio value:
  \[ d\Pi = dV^c_\lambda - \Delta dS^c. \]

- $dS^c$: see previous lemma.
- $dV^c_\lambda$: use Ito’s lemma.

- The portfolio is risk-free if:
  \[ \Delta = \frac{1}{\sigma^c} \frac{\partial V^c_\lambda}{\partial B^c_\lambda}. \]
The PDE for the comonotonic basket derivative price

**Theorem**

The PDE for the derivative price $V^c_\lambda$:

$$
\frac{\partial V^c_\lambda}{\partial t} + \frac{1}{2} \frac{\partial^2 V^c_\lambda}{\partial B^2_\lambda} - rV^c_\lambda = 0,
$$

where the final condition is given by

$$
V^c_\lambda (T, B_\lambda) = H \left( \sum_{i=1}^{n} w_i S_i(0) e^{\left( r - \frac{1}{2} \sigma_i^2 \right) T + \sigma_i B_\lambda} \right).
$$
The PDE in terms of the comonotonic basket:

\[
\frac{\partial V_c}{\partial t'} + \frac{1}{2} \left( \sigma_c(t, B) \right)^2 \frac{\partial^2 V_c}{\partial (S^c)^2} + r S^c \frac{\partial V_c}{\partial S^c} - r V_c = 0.
\]

The final condition:

\[ V_c(T, S^c) = H(S^c). \]

Remarks:

- Similar to the Black & Scholes PDE for one-dimensional derivative pricing.
- **Time-dependent** volatility slows down the calculations.
Theorem

Closed-form solution for $V^c_\lambda (t, B_\lambda)$:

$$V^c_\lambda (t, B_\lambda) = e^{-r(T-t)} \int_{-\infty}^{+\infty} H \left( \sum_{i=1}^{n} w_i S^c_i(t) e^{(r-\frac{1}{2}\sigma_i^2)(T-t)+\sigma_i y} \right) \phi_{T-t}(y) dy,$$  \hspace{1cm} (3)

where $S^c_i(t)$ is given by

$$S^c_i(t) = S_i(0) e^{(r-\frac{1}{2}\sigma_i^2)t+\sigma_i B_\lambda(t)}, \text{ for } i = 1, 2, \ldots, n,$$  \hspace{1cm} (4)

and $\phi_{T-t}$ is the density of a normal distribution with mean 0 and variance $T-t$:

$$\phi_{T-t}(y) = \frac{e^{-\frac{y^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}.$$
3 – The comonotonic market
The solution of the comonotonic PDE

- **Weighted pay-off:**
  - future realization of the increment of the process \( B_\lambda \) in \([t, T] = y\);
  - pay-off: \( H\left(\sum_{i=1}^{n} w_i S_i^c(t)e^{\left(r - \frac{1}{2} \sigma_i^2\right)(T-t)+\sigma_i y}\right)\);
  - Gaussian density in \( y \):
    \[
g(y) = \frac{1}{\sqrt{2\pi(T - t)}} e^{-\frac{y^2}{2(T-t)}}.
\]

- **Conclusion:**
  - We determine the price \( V^c_\lambda(t, B_\lambda) \) by integrating over all future states of the process \( B_\lambda \);
  - weighted by the corresponding Gaussian probabilities.
Theorem

The price $V^c(t, S^c)$ is given by

$$V^c(t, S^c) = e^{-r(T-t)} \int_0^{+\infty} f_{S^c}(S'; T, t) H(S') dS',$$

where

$$f_{S^c}(S'; T, t) = \frac{1}{\sqrt{2\pi(T-t)}} \frac{e^{-\left(\frac{\Phi^{-1}(F_{S^c}(S';T,t))}{2}\right)^2}}{\sum_{i=1}^n \sigma_i w_i S^c_i(t) e^{\left(r - \frac{1}{2} \sigma_i^2\right)(T-t) + \sigma_i \sqrt{T-t} \Phi^{-1}(F_{S^c}(S';T,t))}},'$$

is the time-$t$ risk-neutral density of the comonotonic basket $S^c(T)$ and $F_{S^c}(S'; T, t)$ is the solution of:

$$\sum_{i=1}^n w_i S^c_i(t) e^{\left(r - \frac{1}{2} \sigma_i^2\right)(T-t) + \sigma_i \sqrt{T-t} \Phi^{-1}(F_{S^c}(S';T,t))} = S'.$$
Pricing formulas

- **The price** $V_c^c(t, S^c)$:
  - is an integral over all future realizations of the comonotonic basket $S^c(T)$;
  - weighted using the risk-neutral density;
  - is the discounted risk-neutral expectation.

- **The price** $V_{\lambda}^c(t, B_{\lambda})$:
  - is an integral over all future realizations of the risk factor $B_{\lambda}(T)$;
  - weighted using a Gaussian density;
  - is the solution of a partial differential equation.
A price $V_c^\lambda$ can be obtained by numerically solving the PDE:

$$\frac{\partial V_c^\lambda}{\partial t} + \frac{1}{2} \frac{\partial^2 V_c^\lambda}{\partial B^2_\lambda} - rV_c^\lambda = 0$$

Discretisation:

- Time grid:
  $$t_k = T - k\delta t, \text{ for } k = 0, 1, \ldots, L.$$ 
- Brownian motion grid:
  $$b_j = (j - I)\delta B, \text{ for } j = 0, 1, \ldots, J.$$ 

We determine basket derivative prices on the grid points:

$$V_j^k \equiv V_c^\lambda(t_k, b_j).$$
Basket derivative prices at the maturity end points:

\[ V_j^0 = H \left( \sum_{i=1}^{n} w_i S_i(0) e^{(r - \frac{1}{2} \sigma_i^2) T + \sigma_i b_j} \right). \] (6)

Backwards explicit scheme:

\[
V_j^{k+1} = \frac{1}{2} \frac{\delta t}{\delta B^2} V_j^{k-1} + \left(1 - r\delta t - \frac{\delta t}{\delta B^2}\right) V_j^k + \frac{1}{2} \frac{\delta t}{\delta B^2} V_{j+1}^k + O(\delta t; \delta B^2),
\]

for \( j = 1, \ldots, J - 1 \) and \( k = 1, \ldots, L \).
This scheme is
- easy to implement;
- converges to the real solution.

Stability criterium:
\[ \delta t \leq \frac{2\delta B^2}{r\delta B^2 + 2}. \]

Increasing the accuracy:
- Decrease the step size \( \delta B \);
- Increase the number of points \( J \) in the grid for the Brownian motion.
American-type option:
- The option can be exercised prior to maturity.

Finite difference scheme for pricing American-type derivatives:
- Time grid: $0 = t_L < t_{L-1} < \ldots < t_1 < t_0 = T$.
- Assume: All prices are determined for $t > t_k$.
- Goal: Determine the prices for $t_{k+1}$.

In the interval $[t_{k+1}, t_k]$ there are two possibilities:
- Possibility 1: The derivative is not exercised at time $t_{k+1}$.
- Possibility 2: The derivative is exercised at time $t_{k+1}$. 
Possibility 1: The derivative is not exercised at time $t_{k+1}$.

- The derivative behaves as a European-type derivative in $[t_{k+1}, t_k]$.
- The price $\tilde{V}_{j}^{k+1}$ satisfies:
  $$\tilde{V}_{j}^{k+1} \approx \frac{1}{2} \frac{\delta t}{\delta B^2} V_{j-1}^{k} + \left(1 - r\delta t - \frac{\delta t}{\delta B^2}\right) V_{j}^{k} + \frac{1}{2} \frac{\delta t}{\delta B^2} V_{j+1}^{k}.$$  

Possibility 2: The derivative is exercised at time $t_{k+1}$.

- We receive the payoff:
  $$H \left( S_{j}^{k+1} \right).$$

American-type derivative price in the node $(j, k + 1)$:

$$V_{j}^{k+1} = \max \left\{ \tilde{V}_{j}^{k+1}, H \left( S_{j}^{k+1} \right) \right\}.$$
Approximating the financial market

- **In a comonotonic market:**
  - one risk-factor drives all stocks, the basket and the derivative prices;
  - basket derivative pricing is fast and efficient:
    - closed form expressions;
    - efficient comonotonic finite difference scheme.

- **The real market situation:**
  - the financial market is in general **not** comonotonic;
  - correlations are not equal to one;
  - assumption\(^4\): \(\rho_{i,j} > 0\).

- **Conclusion:**
  
  We didn’t solve the **complete** problem!

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\(^4\)our methodology can be generalized such that this assumption can be relaxed.
Problem:
- Determine the price $V(t)$ of a basket derivative;
- where the components of the basket are correlated.

We construct an artificial financial market:
- the artificial market is required to be comonotonic:
  - closed-form expressions for basket derivative prices are available;
- but: the stocks in the artificial market have an adjusted volatility:
  - the basket derivative price in the artificial market should be ‘close’ to the real price $V$. 
5 – Approximate basket derivative pricing

Approximating the financial market

- **The artificial financial market:**

\[
\frac{dS_i^l(t)}{S_i^l(t)} = \mu_i dt + \nu_i \sigma_i dB(t), \text{ for } t > 0 \text{ and } i = 1, 2, \ldots, n,
\]

where\(^5\)

\[
\nu_i = \frac{\sum_{j=1}^{n} w_j S_j(0) \rho_{i,j} \sigma_j}{\sqrt{\sum_{j=1}^{n} \sum_{k=1}^{n} w_j w_k S_j(0) S_k(0) \rho_{j,k} \sigma_j \sigma_k}}, \text{ for } i = 1, 2, \ldots, n.
\]

- **Remarks:**
  - \(0 < \nu_i \leq 1.\)
  - Adjusted volatility: \(r_i \sigma_i \leq \sigma_i.\)

\(^5\)other choices are possible: Deelstra, Liinev & Vanmaele (2004) and Hainaut & Deelstra (2014).
The marginal stock price process \( \{ S_i^l(t) \mid t \geq 0 \} \)
- are following a Black & Scholes model;
- but: with adjusted volatility parameter.

Dependence = comonotonic copula
- the artificial market is driven by the single Brownian motion \( B \);
- basket derivative pricing is fast and efficient:
  - closed-form solutions are available (single integration);
  - numerical methods are fast and accurate (comonotonic finite difference scheme).
Final approximation

- **Approximate basket:**

  \[ S^l(t) = w_1 S^l_1(t) + w_2 S^l_2(t) + \ldots + w_n S^l_n(t). \]

- **Basket derivative price:**

  \[ V^l(t, S). \]
Table: Input parameters for the four-stock basket with Correlation $\rho = 0.3$.

<table>
<thead>
<tr>
<th></th>
<th>stock 1</th>
<th>stock 2</th>
<th>stock 3</th>
<th>stock 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_i$</td>
<td>0.5</td>
<td>0.2</td>
<td>0.8</td>
<td>0.9</td>
</tr>
<tr>
<td>$S_j(0)$</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>$w_i$</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
</tr>
</tbody>
</table>
Basket put option prices

- Prices $V^{sim}$
- Lower Bound $V^l$
- Upper Bound $V^c$

Strikes:
- 95
- 100
- 105
- 110
- 115

Prices:
- 8
- 10
- 12
- 14
- 16
- 18
- 20
- 22
- 24
- 26
- 28
Figure: The approximation $V^l$ for the basket option price $V$ in function of the spot price $S(0)$ and the time-to-maturity $T$, together with the corresponding $\Delta$ of $V^l$. 
Final approximation for the European basket put option\textsuperscript{6}:

$$\bar{V}(t, S_1, S_2, \ldots, S_n) = z V^l(t, S) + (1 - z) V^c(t, S),$$

where

$$z = \frac{\text{Var}_t [S^c(T)] - \text{Var}_t [S(T)]}{\text{Var}_t [S^c(T)] - \text{Var}_t [S^l(T)]} \in [0, 1].$$

The approximation $\bar{V}$ satisfies:

$$\int_0^\infty \bar{V} dK = \int_0^\infty V dK.$$

\textsuperscript{6}Vyncke, Goovaerts & Dhaene (2004)
Goal: Determine the price of an American-type Basket put option.
- Use Least-squares Monte Carlo Simulation to approximate the real price.
- Use the approximation $\tilde{V}$. 
Table: American-type basket put option prices on a basket of 8 equally weighted stocks with initial prices 40 computed using finite difference (FD) method and the LSM.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Strike</th>
<th>$\sigma_1$</th>
<th>$\rho$</th>
<th>FD prices</th>
<th>LSM prices</th>
<th>time FD</th>
<th>time LSM</th>
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<tbody>
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<td>2</td>
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<td>0.3</td>
<td>4.035</td>
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<td>37</td>
<td>729</td>
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<td>0.8</td>
<td></td>
<td>6.196</td>
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<tr>
<td></td>
<td>0.9</td>
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<td>167</td>
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<td>0.8</td>
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<td>7.477</td>
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<td>40</td>
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<td>6.775</td>
<td>6.704</td>
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<td></td>
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<td>0.8</td>
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<td>9.204</td>
<td>9.176</td>
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<td>10.011</td>
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<td>14.075</td>
<td>14.049</td>
<td>164</td>
<td>4836</td>
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</table>
Computation times in function of the basket size

- Finite difference method
- Least-Squares method

Computing time in sec

Basket size
Table: American-type basket put option prices on a basket of 4 equally weighted stocks with initial prices 40 for high pairwise correlations.

<table>
<thead>
<tr>
<th>Pairwise correlation</th>
<th>FD prices</th>
<th>LSM prices</th>
<th>Comonotonic LSM</th>
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<tr>
<td>0.95</td>
<td>7.427</td>
<td>7.433</td>
<td>–</td>
</tr>
<tr>
<td>0.96</td>
<td>7.452</td>
<td>7.444</td>
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<tr>
<td>0.97</td>
<td>7.476</td>
<td>7.476</td>
<td>–</td>
</tr>
<tr>
<td>0.98</td>
<td>7.500</td>
<td>7.511</td>
<td>–</td>
</tr>
<tr>
<td>0.99</td>
<td>7.524</td>
<td>8.670</td>
<td>–</td>
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<tr>
<td>0.992</td>
<td>7.529</td>
<td>10.250</td>
<td>–</td>
</tr>
<tr>
<td>0.994</td>
<td>7.534</td>
<td>10.234</td>
<td>–</td>
</tr>
<tr>
<td>0.996</td>
<td>7.538</td>
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<td>0.998</td>
<td>7.543</td>
<td>17.212</td>
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<td>1.000</td>
<td>7.548</td>
<td>259.274</td>
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Thank you for your attention!