Natural hedges with immunization strategies of mortality and interest rates

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Durations and convexity for interest (yield)

In finance, dollar duration measures the sensitivity of the price of an asset or liability with respect to a constant change in the interest rate, and convexity measures the curvature or the second derivative of the price.

Let $P(\delta) = \sum_{k=1}^{n} C_k \times e^{-\delta \cdot k}$ be the price of a financial security at time 0 with cash flows $C_k$ at time $k$, $k = 1, \ldots, n$.

- **Dollar Duration** $DD_\delta [P(\delta)] = -\frac{\partial P(\delta)}{\partial \delta} = \sum_{k=1}^{n} k \times C_k \times e^{-\delta \cdot k}$

- **Dollar Convexity** $DC_\delta [P(\delta)] = \frac{\partial^2 P(\delta)}{\partial \delta^2} = \sum_{k=1}^{n} k^2 \times C_k \times e^{-\delta \cdot k}$

Macaulay duration $MD_\delta [P(\delta)] = DD_\delta [P(\delta)]/(\delta) = -\frac{d P(\delta)/P(\delta)}{d \delta}$ measures the sensitivity of the yield (rate of price change) with respect to a constant change in the interest rate.
**Immunization of interest rate risk**

Figure: cash flows \( \{L_k : k = 1, \ldots, n\} \) and \( \{A_k : k = 1, \ldots, n\} \)

\[
L(\delta) = \sum_{k=1}^{n} L_k \times e^{-\delta \cdot k} \quad \text{and} \quad A(\delta) = \sum_{k=1}^{n} A_k \times e^{-\delta \cdot k}
\]
at time 0, where \( \delta = \ln(1 + i) \) is the force of interest and \( i \) is the interest rate.

Interest rate immunization: both \( L(\delta) \) and \( A(\delta) \) change to \( L(\delta + \gamma) \) and \( A(\delta + \gamma) \) when \( \delta \) shifts to \( \delta + \gamma \).

We want to allocate \( \{A_k : k = 1, \ldots, n\} \) such that

\[
A(\delta + \gamma) - A(\delta) = \Delta A(\gamma) \approx \Delta L(\gamma) = L(\delta + \gamma) - L(\delta),
\]
that is, the interest rate risk is immunized.
**Duration matching**

\[
\Delta A_{\gamma}(\delta) = \frac{A(\delta + \gamma) - A(\delta)}{\gamma} \cdot \gamma \approx \frac{\partial A(\delta)}{\partial \delta} \cdot \gamma,
\]

\[
\Delta L_{\gamma}(\delta) = \frac{L(\delta + \gamma) - L(\delta)}{\gamma} \cdot \gamma \approx \frac{\partial L(\delta)}{\partial \delta} \cdot \gamma,
\]

\[
\Delta A_{\gamma}(\delta) = \Delta L_{\gamma}(\delta) \Rightarrow \frac{\partial A(\delta)}{\partial \delta} = \frac{\partial L(\delta)}{\partial \delta},
\]

that is, dollar duration of \( A(\delta) = \) dollar duration of \( L(\delta) \). Then the interest rate risk of the liabilities is immunized by duration matching.
**Effective Duration and Convexity**

Let \( P(\delta) = \sum_{k=1}^{n} C_k \times e^{-\delta \cdot k} \) and \( P(\tilde{\delta}) = \sum_{k=1}^{n} C_k \times e^{-\sum_{i=0}^{k-1} \delta_i} \) where \( \delta_i \) is the force of interest for the \((i + 1)^{th}\) year or the period \([i, i + 1]\) and \( \tilde{\delta} = (\delta_0, \delta_1, \ldots, \delta_{n-1}) \).

- Dollar duration \( DD_{\delta}[P(\delta)] = -\frac{\partial P(\delta)}{\partial \delta} \)
- Dollar convexity \( DC_{\delta}[P(\delta)] = \frac{\partial^2 P(\delta)}{\partial \delta^2} \)
- Effective dollar duration \( EDD_{\gamma}[P(\tilde{\delta})] = -\frac{P(\tilde{\delta} + \gamma) - P(\tilde{\delta} - \gamma)}{2 \cdot \gamma} \)
- Effective dollar convexity

\[
EDC_{\gamma}[P(\tilde{\delta})] = \frac{P(\tilde{\delta} + \gamma) + P(\tilde{\delta} - \gamma) - 2 \cdot P(\delta)}{\gamma^2} - \frac{P(\tilde{\delta} + \gamma) - P(\delta) - P(\delta) - P(\tilde{\delta} - \gamma)}{\gamma}
\]
Alternative derivation for dollar duration/convexity

\[ P(\delta) = \sum_{k=1}^{n} C_k \times e^{-\delta \cdot k} = \sum_{k=1}^{n} C_k \times f_k(\delta) \] where \( f_k(\delta) = e^{-\delta \cdot k} \).

When \( \delta \to \delta + \gamma \), then \( f_k(\delta) \to f_k(\delta + \gamma) = f_k(\delta) \cdot f_k(\gamma) \), and
\[ \Delta \gamma f_k(\delta) = f_k(\delta + \gamma) - f_k(\delta) = f_k(\delta) \cdot [f_k(\gamma) - 1]. \]

Expand \( f_k(\gamma) \) to \( f_k(\gamma) \approx 1 + \frac{\partial f_k(\gamma)}{\partial \gamma} \bigg|_{\gamma=0} \cdot \gamma + \frac{\partial^2 f_k(\gamma)}{\partial \gamma^2} \bigg|_{\gamma=0} \cdot \frac{\gamma^2}{2} \).

\[ \Delta \gamma f_k(\delta) \approx f_k(\delta) \cdot [-k \cdot \gamma + k^2 \cdot \gamma^2 / 2] = e^{-\delta \cdot k} \cdot [-k \cdot \gamma + k^2 \cdot \gamma^2 / 2]. \]

\[ \Delta \gamma P(\delta) = P(\delta + \gamma) - P(\delta) = \sum_{k=1}^{n} C_k \cdot \Delta \gamma f_k(\delta) \]
\[ \approx \sum_{k=1}^{n} C_k \cdot e^{-\delta \cdot k} \cdot [-k \cdot \gamma + k^2 \cdot \gamma^2 / 2] \]
\[ \approx -\left[ \sum_{k=1}^{n} k \cdot C_k \cdot e^{-\delta \cdot k} \right] \cdot \gamma + \left[ \sum_{k=1}^{n} k^2 \cdot C_k \cdot e^{-\delta \cdot k} \right] \cdot \frac{\gamma^2}{2} \]
\[ = -DD_\delta [P(\delta)] \cdot \gamma + DC_\delta [P(\delta)] \cdot \gamma^2 / 2. \]
Dollar duration/convexity w.r.t. a constant change in $\delta$

$P(\tilde{\delta}) = \sum_{k=1}^{n} C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i}$ = $\sum_{k=1}^{n} C_k \cdot f_k(\tilde{\delta})$ where $f_k(\tilde{\delta}) = e^{-\sum_{i=0}^{k-1} \delta_i}$.

When $\tilde{\delta} \to \tilde{\delta} + \gamma$, then $f_k(\tilde{\delta}) \to f_k(\tilde{\delta} + \gamma) = f_k(\tilde{\delta}) \cdot f_k(\gamma)$ where $f_k(\gamma) = e^{-\gamma \cdot k}$, and $\Delta^c f_k(\tilde{\delta}) = f_k(\tilde{\delta} + \gamma) - f_k(\tilde{\delta}) = f_k(\tilde{\delta}) \cdot [f_k(\gamma) - 1]$.

Expand $f_k(\gamma)$ to $f_k(\gamma) \approx 1 + \frac{\partial f_k(\gamma)}{\partial \gamma} \bigg|_{\gamma=0} \cdot \gamma + \frac{\partial^2 f_k(\gamma)}{\partial \gamma^2} \bigg|_{\gamma=0} \cdot \gamma^2 / 2$.

$\Delta^c f_k(\tilde{\delta}) \approx f_k(\tilde{\delta}) \cdot [-k \cdot \gamma + k^2 \cdot \gamma^2 / 2] = e^{-\sum_{i=0}^{k-1} \delta_i} \cdot [-k \cdot \gamma + k^2 \cdot \gamma^2 / 2]$.

$\Delta^c P(\tilde{\delta}) = P(\tilde{\delta} + \gamma) - P(\tilde{\delta}) = \sum_{k=1}^{n} C_k \cdot \Delta^c f_k(\tilde{\delta})$

$\approx \sum_{k=1}^{n} C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i} \cdot [-k \cdot \gamma + k^2 \cdot \gamma^2 / 2]$.

$\approx -\left[\sum_{k=1}^{n} k \cdot C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i}\right] \cdot \gamma + \left[\sum_{k=1}^{n} k^2 \cdot C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i}\right] \cdot \gamma^2 / 2$

$= -DD^c_\delta[P(\tilde{\delta})] \cdot \gamma + DC^c_\delta[P(\tilde{\delta})] \cdot \gamma^2 / 2.$
**Dollar duration/convexity w.r.t. a proportional shift in \( \tilde{\delta} \)**

\[ P(\tilde{\delta}) = \sum_{k=1}^{n} C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i} = \sum_{k=1}^{n} C_k \cdot f_k(\tilde{\delta}) \]  
where  
\[ f_k(\tilde{\delta}) = e^{-\sum_{i=0}^{k-1} \delta_i}. \]

When \( \tilde{\delta} \rightarrow (1 + \gamma) \cdot \tilde{\delta}, \) then  
\[ f_k(\tilde{\delta}) \rightarrow f_k((1 + \gamma) \cdot \tilde{\delta}) = [f_k(\tilde{\delta})]^{1+\gamma}, \] and  
\[ \Delta_p^\gamma f_k(\tilde{\delta}) = f_k((1 + \gamma) \cdot \tilde{\delta}) - f_k(\tilde{\delta}) = f_k(\tilde{\delta}) \cdot \{[f_k(\tilde{\delta})]^\gamma - 1\}. \]

Expand \([f_k(\tilde{\delta})]^\gamma\) to \([f_k(\tilde{\delta})]^\gamma \approx 1 + \frac{\partial [f_k(\tilde{\delta})]^\gamma}{\partial \gamma} \biggr|_{\gamma=0} \cdot \gamma + \frac{\partial^2 [f_k(\tilde{\delta})]^\gamma}{\partial \gamma^2} \biggr|_{\gamma=0} \cdot \frac{\gamma^2}{2}. \]

\[ \Delta_p^\gamma f_k(\tilde{\delta}) \approx e^{-\sum_{i=0}^{k-1} \delta_i} \cdot \left\{ -\left[\sum_{i=0}^{k-1} \delta_i\right] \cdot \gamma + \left[\sum_{i=0}^{k-1} \delta_i\right]^2 \cdot \frac{\gamma^2}{2} \right\}. \]

\[ \Delta^\gamma_p P(\tilde{\delta}) = P((1 + \gamma) \cdot \tilde{\delta}) - P(\tilde{\delta}) = \sum_{k=1}^{n} C_k \cdot \Delta^\gamma_p f_k(\tilde{\delta}) \]

\[ \approx \sum_{k=1}^{n} C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i} \cdot \left\{ -\left[\sum_{i=0}^{k-1} \delta_i\right] \cdot \gamma + \left[\sum_{i=0}^{k-1} \delta_i\right]^2 \cdot \frac{\gamma^2}{2} \right\} \]

\[ \approx -\left[\sum_{k=1}^{n} \left(\sum_{i=0}^{k-1} \delta_i\right) \cdot C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i}\right] \cdot \gamma + \left[\sum_{k=1}^{n} \left(\sum_{i=0}^{k-1} \delta_i\right)^2 \cdot C_k \cdot e^{-\sum_{i=0}^{k-1} \delta_i}\right] \cdot \frac{\gamma^2}{2} \]

\[ = -DD^p_\delta[P(\tilde{\delta})] \cdot \gamma + DC^p_\delta[P(\tilde{\delta})] \cdot \gamma^2/2. \]
The net single premium of a more general annuity product - the $h$-year deferred and $j$-year life annuity-due, issued to $(x)$ in year $t$, assuming piecewise constant force of mortality \( (\mu_x, t(s) = \mu_{x+i}, t+i, s \in [i, i+1]) \),

\[
\begin{align*}
    h|\ddot{a}_x, t: j| &= \sum_{k=h}^{h+j-1} kp_x, t \cdot e^{-\delta \cdot k} = \sum_{k=h}^{h+j-1} e^{-\int_0^k \mu_x, t(s) \, ds} \cdot e^{-\delta \cdot k} = \\
    &= \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1} \mu_{x+i}, t+i} \cdot e^{-\delta \cdot k},
\end{align*}
\]

where \( kp_x, t = p_x, t \times \cdots \times p_{x+k-1}, t+k-1 \).

- \( (h, j) = (0, n) \Rightarrow \ddot{a}_x, t: \overline{n} \),
- \( (h, j) = (n, 1) \Rightarrow nEx, t \),
- \( (h, j) = (n, \infty) \Rightarrow n|\ddot{a}_x, t \),
- \( (h, j) = (0, \infty) \Rightarrow \ddot{a}_x, t \).

The NSP of life insurance can be expressed in terms of NSPs of \( h|\ddot{a}_x, t: j| \):

- \( A_x, t: \overline{n} = 1 - d \cdot \ddot{a}_x, t: \overline{n} \),
- \( A_x, t = 1 - d \cdot \ddot{a}_x, t \),
- \( A^1_x, t: \overline{n} = A_x, t: \overline{n} - nEx, t = 1 - d \cdot \ddot{a}_x, t: \overline{n} - nEx, t \).
Relational models

Let $\tilde{\mu}_{x,t} = \{\mu_{x,t}(i) = \mu_{x+i,t+i} : i = 0, 1, \cdots\}$ be a force of morality sequence, starting age $x$ in year $t$.

Linear relational model (Tsai and Yang (2015))

$$\tilde{\mu}_{x,t}^B = (1 + \alpha) \times \tilde{\mu}_{x,t}^A + \beta + e_{x,t}.$$ 

When $\tilde{\mu}_{x,t}$ is shifted proportionally to $(1 + \alpha) \cdot \tilde{\mu}_{x,t}$ and moved constantly to $\tilde{\mu}_{x,t}^\bullet = (1 + \alpha) \cdot \tilde{\mu}_{x,t} + \beta$, then $k p_{x,t}$ is changed to

$$kp_{x,t}^\bullet = kp_{x,t} \cdot \left( kp_{x,t} \right)^\alpha \cdot e^{-\beta \cdot k} = kp_{x,t} \cdot f_{\tilde{\mu}_{x,t}}^p(\alpha) \cdot f_{\tilde{\mu}_{x,t}}^c(\beta).$$

- $\beta = 0$ (a proportional change only): $kp_{x,t}^\bullet = kp_{x,t} \cdot f_{\tilde{\mu}_{x,t}}^p(\alpha)$ and $\Delta kp_{x,t} = kp_{x,t}^\bullet - kp_{x,t} = kp_{x,t} \cdot [f_{\tilde{\mu}_{x,t}}^p(\alpha) - 1]$ with $f_{\tilde{\mu}_{x,t}}^p(0) = 1$.
- $\alpha = 0$ (a constant change only): $kp_{x,t}^\bullet = kp_{x,t} \cdot f_{\tilde{\mu}_{x,t}}^c(\beta)$ and $\Delta kp_{x,t} = kp_{x,t}^\bullet - kp_{x,t} = kp_{x,t} \cdot [f_{\tilde{\mu}_{x,t}}^c(\beta) - 1]$ with $f_{\tilde{\mu}_{x,t}}^c(0) = 1$. 

\( \tilde{\mu}_{30, 1989+n} \) AGAINST \( \tilde{\mu}_{30, 1989} \) FOR UK MALES
A realized/simulated $\tilde{\mu}_{25,2010}$ against the expected $\tilde{\mu}_{25,2010}$ for US males
Taylor's expansion of \( f_{\tilde{\mu}_x, t}(\gamma), (\lambda, \gamma) = (p, \alpha), (c, \beta) \)

\[
f_{\tilde{\mu}_x, t}(\gamma) \approx f_{\tilde{\mu}_x, t}(0) + \left. \frac{\partial f_{\tilde{\mu}_x, t}(\gamma)}{\partial \gamma} \right|_{\gamma=0} \cdot \gamma + \left. \frac{\partial^2 f_{\tilde{\mu}_x, t}(\gamma)}{\partial \gamma^2} \right|_{\gamma=0} \cdot \frac{\gamma^2}{2}. \]

\[
\triangle_{kp_x, t} = kp_x, t \cdot [f_{\tilde{\mu}_x, t}(\gamma) - 1] \approx kp_x, t [d\lambda(k) \cdot \gamma + c\lambda(k) \cdot \gamma^2/2], \text{ where}
\]

\[
d\lambda(k) = \left[ \frac{\partial f_{\tilde{\mu}_x, t}(\gamma)}{\partial \gamma} \right]_{\gamma=0} = \begin{cases} \ln(kp_x, t), & \lambda = p, \\ -k, & \lambda = c; \end{cases} \text{ and}
\]

\[
c\lambda(k) = \left[ \frac{\partial^2 f_{\tilde{\mu}_x, t}(\gamma)}{\partial \gamma^2} \right]_{\gamma=0} = \begin{cases} [\ln(kp_x, t)]^2, & \lambda = p, \\ k^2, & \lambda = c. \end{cases} = [d\lambda(k)]^2.
\]

\[
\Delta_h|\ddot{a}_{x, t: j}| = h|\dddot{a}_{x, t: j}| - h|\dddot{a}_{x, t: j}| = \sum_{k=h}^{h+j-1} \triangle_{kp_x, t} \cdot e^{-\delta \cdot k}
\]

\[
\approx D\lambda[h|\dddot{a}_{x, t: j}|(\tilde{\mu}_x, t)] \cdot \gamma + C\lambda[h|\dddot{a}_{x, t: j}|(\tilde{\mu}_x, t)] \cdot \gamma^2/2, \text{ where}
\]

\[
B^\lambda[h|\dddot{a}_{x, t: j}|(\tilde{\mu}_x, t)] = \sum_{k=h}^{h+j-1} b\lambda(k) \cdot kp_x, t \cdot e^{-\delta \cdot k} \text{ is called mortality duration for (B, b) = (D, d) (convexity for (B, b) = (C, c)) w.r.t. an instantaneously proportional (}\lambda = p\) \text{ or constant (}\lambda = c\) \text{ change in } \tilde{\mu}_x, t.
\]


**Insurance Portfolio** $P^{LA}$ of Two Policies

$P^{LA}$: discrete life insurance and an annuity with weights $w_L$ and $w_A = 1 - w_L$, respectively; the weighted surplus (negative reserve) at time 0 is

$$0S_{x, t}^{LA} = w_L \cdot 0S_{x_l}^{PL} + (1 - w_L) \cdot 0S_{x_a}^{DA} = 0$$

- $0S_{x_l}^{PL} = m(\tilde{\mu}_{x_l}, t) - A_{x_l}(\tilde{\mu}_{x_l}, t) = 0$ is the surplus at time 0 for a discrete $m$-payment whole life insurance;
- $0S_{x_a}^{DA} = n(\tilde{\mu}_{x_a}, t) - \tilde{\mu}_{x_a}(\tilde{\mu}_{x_a}, t) = 0$ is the surplus at time 0 for an $n$-year deferred whole life annuity-due;

When both $\tilde{\mu}_{x_l}, t$ and $\tilde{\mu}_{x_a}, t$ are shifted proportionally by $\gamma$ or moved constantly by $\gamma$, the changes in $0S_{x, t}^{LA}$, $0S_{x_l}^{PL}$, and $0S_{x_a}^{DA}$ are

$$\triangle \lambda 0S_{x, t}^{LA} = w_L \cdot \triangle \lambda 0S_{x_l}^{PL} + (1 - w_L) \cdot \triangle \lambda 0S_{x_a}^{DA}$$

$$\triangle \lambda 0S_{x_l}^{PL} = D^{\lambda}[0S_{x_l}^{PL}(\tilde{\mu}_{x_l}, t)] \cdot \gamma + C^{\lambda}[0S_{x_l}^{PL}(\tilde{\mu}_{x_l}, t)] \cdot \gamma^2 / 2,$$

$$\triangle \lambda 0S_{x_a}^{DA} = D^{\lambda}[0S_{x_a}^{DA}(\tilde{\mu}_{x_a}, t)] \cdot \gamma + C^{\lambda}[0S_{x_a}^{DA}(\tilde{\mu}_{x_a}, t)] \cdot \gamma^2 / 2.$$
Matching strategy $B^\lambda(\bar{\mu}_x, t)$ based on $\hat{w}_L$

$$\triangle^\lambda S^L_{x,t} = w_L \cdot \triangle^\lambda S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t) + (1 - w_L) \cdot \triangle^\lambda S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)$$

$$\approx \{ w_L \cdot D^\lambda [0 S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t)] + (1 - w_L) \cdot D^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)] \} \gamma$$

$$+ \{ w_L \cdot C^\lambda [0 S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t)] + (1 - w_L) \cdot C^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)] \} \frac{\gamma^2}{2}.$$ 

Find $\hat{w}_L$ such that $\triangle^\lambda S^L_{x,t} \approx 0$ (that is, the life insurance portfolio is immunized with respect to a change in mortality rates).

$$w^*_L \cdot B^\lambda [0 S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t)] + (1 - w_L) \cdot B^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)] = 0$$

$$\Rightarrow \hat{w}_L = \frac{B^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)]}{B^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)] - B^\lambda [0 S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t)]} \cdot \frac{1}{1 - B^\lambda [0 S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t)]/B^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)]},$$

where

$$B^\lambda [0 S^{PL}_{x_l, t} : m(\bar{\mu}_{x_l}, t)] = P^{PL}_{x_l, t} : m \cdot B^\lambda [\bar{a}_{x_l, t} : \bar{m}](\bar{\mu}_{x_l}, t) - B^\lambda [A_{x_l, t} : (\bar{\mu}_{x_l}, t)],$$

$$B^\lambda [0 S^{DA}_{x_a, t} : n(\bar{\mu}_{x_a}, t)] = P^{DA}_{x_a, t} : n \cdot B^\lambda [\bar{a}_{x_a, t} : \bar{n}](\bar{\mu}_{x_a}, t) - B^\lambda [n \bar{a}_{x_a, t} : (\bar{\mu}_{x_a}, t)],$$

and $B^\lambda = D^\lambda (C^\lambda)$ for mortality duration (convexity) matching w.r.t. an instantaneously proportional ($\lambda = p$) or constant ($\lambda = c$) change in $\bar{\mu}_x, t$. 


**Insurance Portfolio** $P^{LA}$ **of Multiple Policies**

$P^{LA}$: discrete life insurance and an annuity with weights $w_L$ and $w_A = 1 - w_L$, respectively; the weighted surplus (negative reserve) at time 0 is

$$0S_{x,t}^{LA} = w_L \cdot \sum_{x \in X_l} p_x^{PL} \cdot 0S_{x,t}^{PL} \cdot m(\tilde{\mu}_x, t) + (1 - w_L) \cdot \sum_{x \in X_a} p_x^{DA} \cdot 0S_{x,t}^{DA} \cdot n(\tilde{\mu}_x, t) = 0,$$

where $p_x^{PL}$ and $p_x^{DA}$ are the percentages or numbers of life and annuity policyholders aged $x$, respectively.

$$\hat{W}_L^* = \frac{B^\lambda [\sum_{x \in X_a} p_x^{DA} \cdot 0S_{x,t}^{DA} \cdot n(\tilde{\mu}_x, t)]}{B^\lambda [\sum_{x \in X_a} p_x^{DA} \cdot 0S_{x,t}^{DA} \cdot n(\tilde{\mu}_x, t)] - B^\lambda [\sum_{x \in X_l} p_x^{PL} \cdot 0S_{x,t}^{PL} \cdot m(\tilde{\mu}_x, t)]},$$

where

$$B^\lambda [\sum_{x \in X_a} p_x^{DA} \cdot 0S_{x,t}^{DA} \cdot n(\tilde{\mu}_x, t)] = \sum_{x \in X_a} p_x^{DA} \cdot B^\lambda [0S_{x,t}^{DA} \cdot n(\tilde{\mu}_x, t)],$$

$$B^\lambda [\sum_{x \in X_l} p_x^{PL} \cdot 0S_{x,t}^{PL} \cdot m(\tilde{\mu}_x, t)] = \sum_{x \in X_l} p_x^{PL} \cdot B^\lambda [0S_{x,t}^{PL} \cdot m(\tilde{\mu}_x, t)],$$

by linearity preservation property of $B^\lambda$. 
**Force of mortality-interest** \( \mu_{x,t}(i) = \mu_{x,t}(i) + \delta_i \)

\[
\begin{align*}
\dot{h \mid a_x, t: j} &= \sum_{k=h}^{h+j-1} k p_{x,t} \cdot e^{-\int_0^k \delta_s \, ds} = \sum_{k=h}^{h+j-1} e^{-\int_0^k \mu_{x,t}(s) \, ds} \cdot e^{-\int_0^k \delta_s \, ds} \\
&= \sum_{k=h}^{h+j-1} e^{-\int_0^k \mu_{x,t}(s) + \delta_s \, ds} = \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1} \mu_{x,t}(i) + \delta_i} \\
&= \sum_{k=h}^{h+j-1} e^{-\sum_{i=0}^{k-1} \mu_{x,t}(i)} = \sum_{k=h}^{h+j-1} k p_{x,t}^{*},
\end{align*}
\]

where \( \mu_{x,t}(i) = \mu_{x,t}(i) + \delta_i \).

\( B^\lambda[h \mid a_x, t: j](\tilde{\mu}_{x,t}) = \sum_{k=h}^{h+j-1} b^\lambda(k) \cdot k p_{x,t}^{*} \) is called mortality-interest duration for \((B, b) = (D, d)\) (convexity for \((B, b) = (C, c)\)) w.r.t. an instantaneously proportional \(\lambda = p\) or constant \(\lambda = c\) change in \(\tilde{\mu}_{x,t}\),

where \(d^\lambda(k) = \left\{ \begin{array}{ll} 
\ln(k p_{x,t}^{*}), & \lambda = p, \\
-k, & \lambda = c;
\end{array} \right. \) and \(c^\lambda(k) = [d^\lambda(k)]^2\).
A realized/simulated $\tilde{\mu}_{x,t}^*$ vs the expected $\tilde{\mu}_{x,t}^*$

**Figure:** Left: $x = 50$; Right: $x = 60$

Matching strategy $B^\lambda(\tilde{\mu}^*_x, t)$ based on $\hat{w}_L^*$

\[
\hat{w}_L^* = \frac{B^\lambda \left[ \sum_{x \in X_a} p_x^{DA} \cdot 0 S_x^{DA} \cdot n(\tilde{\mu}^*_x, t) \right]}{B^\lambda \left[ \sum_{x \in X_a} p_x^{DA} \cdot 0 S_x^{DA} \cdot n(\tilde{\mu}^*_x, t) \right] - B^\lambda \left[ \sum_{x \in X_l} p_x^{PL} \cdot 0 S_x^{PL} \cdot m(\tilde{\mu}^*_x, t) \right]},
\]

where $B^\lambda = D^\lambda \ (C^\lambda)$ for mortality-interest duration (convexity) matching w.r.t. an instantaneously proportional ($\lambda = p$) or constant ($\lambda = c$) change in $\tilde{\mu}^*_x, t$.

Six matching strategies for comparison:

- $D^p(\tilde{\mu}^*_x, t)$ with $d^*_p(k) = \ln(k p_x^*, t)$,
- $D^c(\tilde{\mu}^*_x, t) (= D^c(\tilde{\mu}^*_x, t))$ with $d^*_c(k) = d_c(k) = -k$,
- $C^p(\tilde{\mu}^*_x, t)$ with $c^*_p(k) = [\ln(k p_x^*, t)]^2$,
- $C^c(\tilde{\mu}^*_x, t) (= C^c(\tilde{\mu}^*_x, t))$ with $c^*_c(k) = c_c(k) = k^2$,
- $D^p(\tilde{\mu}^*_x, t)$ with $d_p(k) = \ln(k p_x, t)$, and
- $C^p(\tilde{\mu}^*_x, t)$ with $c_p(k) = [\ln(k p_x, t)]^2$. 
**Two insurance portfolios and \( HE(\sigma^2) \)**

Two insurance portfolios \( 20P^PL_x \) and \( 65−xP^DA_x \):

- \( 20P^PL_x \): one-unit discrete 20-payment whole life insurance issued to \((x)\) \((x = 35, 40, 45, 50, 55, 60)\) using US male mortality data, and
- \( 65−xP^DA_x \): one-unit \((65x)\)-payment \((65x)\)-year deferred whole life annuity-due issued to \((x)\) using US female mortality data.

\( p_{x, gender} \), percentages (%) of age and gender for two portfolios

<table>
<thead>
<tr>
<th>Age ( x )</th>
<th>35</th>
<th>40</th>
<th>45</th>
<th>50</th>
<th>55</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>PL (Male)</td>
<td>8.39</td>
<td>7.76</td>
<td>8.27</td>
<td>8.44</td>
<td>8.64</td>
<td>7.76</td>
</tr>
<tr>
<td>DA (Female)</td>
<td>8.37</td>
<td>7.85</td>
<td>8.40</td>
<td>8.68</td>
<td>9.06</td>
<td>8.37</td>
</tr>
</tbody>
</table>

Hedge effectiveness w.r.t. variance for mortality risk \( HE_M \) and longevity risk \( HE_L \) are defined as the variance reduction ratio. Specifically,

\[
HE_M(\sigma^2) = \frac{\sigma^2(S^L_0) - \sigma^2(S^P_0)}{\sigma^2(S^L_0)} = 1 - \frac{\sigma^2(S^P_0)}{\sigma^2(S^L_0)} ,
\]

\[
HE_L(\sigma^2) = \frac{\sigma^2(S^A_0) - \sigma^2(S^P_0)}{\sigma^2(S^A_0)} = 1 - \frac{\sigma^2(S^P_0)}{\sigma^2(S^A_0)}. 
\]
## Summary for simulated surpluses (10,000 policyholders)

### Panel A: Two single-product portfolios

<table>
<thead>
<tr>
<th></th>
<th>PL</th>
<th>DA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{w}_L$</td>
<td>1.0000 (largest)</td>
<td>0.0000 (smallest)</td>
</tr>
<tr>
<td>mean</td>
<td>301 (smallest)</td>
<td>6209 (largest)</td>
</tr>
<tr>
<td>std</td>
<td>677 (smallest)</td>
<td>13305 (largest)</td>
</tr>
<tr>
<td>5%-VaR</td>
<td>472 (smallest)</td>
<td>8518 (largest)</td>
</tr>
<tr>
<td>Pr(gain) (%)</td>
<td>57.40</td>
<td>58.10</td>
</tr>
</tbody>
</table>

### Panel B: Six matched portfolios

<table>
<thead>
<tr>
<th></th>
<th>$D^c(\tilde{\mu})$</th>
<th>$D^p(\tilde{\mu})$</th>
<th>$C^c(\tilde{\mu})$</th>
<th>$C^p(\tilde{\mu})$</th>
<th>$D^p(\tilde{\mu}^*)$</th>
<th>$C^p(\tilde{\mu}^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{w}_{L}^*$</td>
<td>0.9700 &lt; 0.9812 &lt; 0.9854 &lt; 0.9876 &lt; 0.9914 &lt; 0.9999</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$HE_M$ (%)</td>
<td>−140.69 &lt; −80.82 &lt; −60.85 &lt; −50.56 &lt; −33.87 &lt; −0.42</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>mean</td>
<td>478 &gt; 412 &gt; 387 &gt; 374 &gt; 351 &gt; 301</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>std</td>
<td>1050 &gt; 910 &gt; 858 &gt; 830 &gt; 783 &gt; 678</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%-VaR</td>
<td>710 &gt; 622 &gt; 588 &gt; 570 &gt; 540 &gt; 473</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pr(gain) (%)</td>
<td>57.79 57.70 57.74 57.76 57.57 57.41</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$D^c(\tilde{\mu}) < D^p(\tilde{\mu}) < C^c(\tilde{\mu}) < C^p(\tilde{\mu}) < D^p(\tilde{\mu}^*) < C^p(\tilde{\mu}^*)$ applies to $\hat{w}_L$, $HE_M$ and $HE_L$; and its reversed order applies to mean, standard deviation and 5%-VaR.
**Adjusted Formula for Weight \( \hat{w}_L^* \)**

\[
\hat{w}_L^* = \frac{B^\lambda [\sum_{x \in x_a} p_x^{DA} \cdot 0S_x^{DA}(\tilde{\mu}^*_x, t)]}{B^\lambda [\sum_{x \in x_a} p_x^{DA} \cdot 0S_x^{DA}(\tilde{\mu}^*_x, t)] - B^\lambda [\sum_{x \in x_l} p_x^{PL} \cdot 0S_x^{PL}(\tilde{\mu}^*_x, t)]}
\]

is based on $1 sum assumed for life insurance and $1 annual payment for all life annuities. If the sum assumed is $SA for life insurance and the annual payment is $AP for all life annuities, then the formula for weight \( \hat{w}_L^* \) is adjusted to

\[
\hat{w}_L^* = \frac{AP \cdot B^\lambda [\sum_{x \in x_a} p_x^{DA} \cdot 0S_x^{DA}(\tilde{\mu}^*_x, t)]}{AP \cdot B^\lambda [\sum_{x \in x_a} p_x^{DA} \cdot 0S_x^{DA}(\tilde{\mu}^*_x, t)] - SA \cdot B^\lambda [\sum_{x \in x_l} p_x^{PL} \cdot 0S_x^{PL}(\tilde{\mu}^*_x, t)]} = \frac{1}{1 - \frac{SA}{AP} \cdot B^\lambda [\sum_{x \in x_l} p_x^{PL} \cdot 0S_x^{PL}(\tilde{\mu}^*_x, t)] / B^\lambda [\sum_{x \in x_a} p_x^{DA} \cdot 0S_x^{DA}(\tilde{\mu}^*_x, t)]}
\]

If \( SA = 300,000 \) and \( AP = 30,000 \) such that \( SA/AP = 10 \) then

<table>
<thead>
<tr>
<th>( SA/AP )</th>
<th>( D^c(\tilde{\mu}) )</th>
<th>( D^p(\tilde{\mu}) )</th>
<th>( C^c(\tilde{\mu}) )</th>
<th>( C^p(\tilde{\mu}) )</th>
<th>( D^p(\tilde{\mu}^*) )</th>
<th>( C^p(\tilde{\mu}^*) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9700 &lt;</td>
<td>0.9812 &lt;</td>
<td>0.9854 &lt;</td>
<td>0.9876 &lt;</td>
<td>0.9914 &lt;</td>
<td>0.9999</td>
</tr>
<tr>
<td>10</td>
<td>0.7641 &lt;</td>
<td>0.8394 &lt;</td>
<td>0.8708 &lt;</td>
<td>0.8886 &lt;</td>
<td>0.9204 &lt;</td>
<td>0.9989</td>
</tr>
</tbody>
</table>
Conclusions

- We propose mortality-interest duration and convexity with respect to an instantaneously proportional or constant change in the force of mortality-interest for the first time in the literature.
- Nature hedge with mortality-interest duration and convexity matching strategies can be applied to a portfolio of life and annuity products issued to a group of life insureds and a group of annuitants with different sizes.
- Mortality-interest duration and convexity matching strategies outperform mortality duration and convexity matching strategies in hedging mortality and longevity risks.
The End

Thank You