A Bayesian approach to modeling mortality rates

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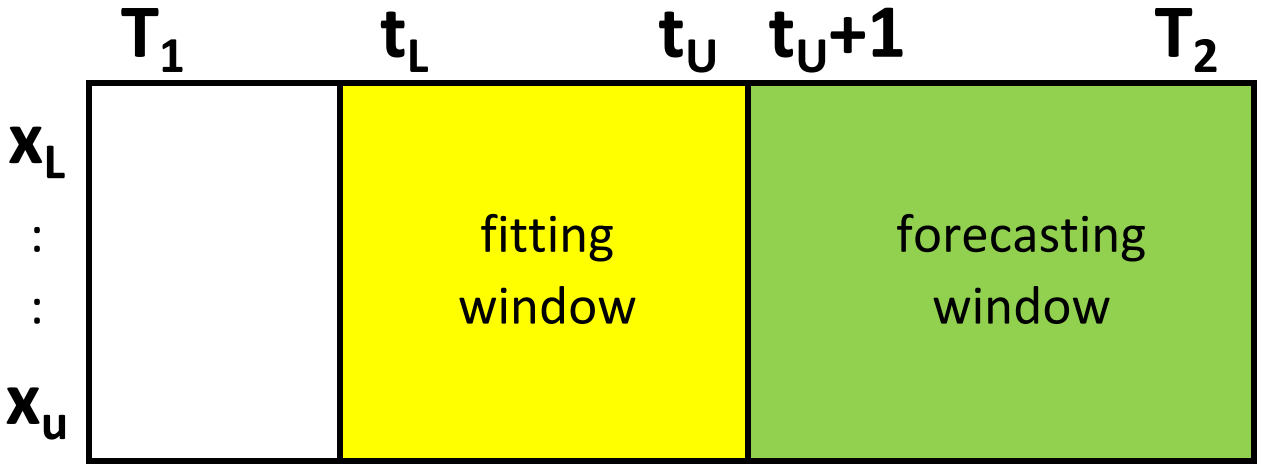
Outline

Motivations of developing effective mortality models

- Modeling the changes and dynamics of mortality rates is critical to the solvency of life insurers and social benefit programs.
- Mortality is also one of the key factors in pricing and reserving of life insurance and annuity products.
- Failing in capturing the downward trends in mortality rates would under-price/-reserve annuity products and then expose annuity providers and social security systems to the risk of financial insolvency.

Fitting window and forecasting window



- $q_{x,t}$ ($p_{x,t} = 1 - q_{x,t}$): one-year death (survival) probability, and $\mu_{x,t}$ ($m_{x,t}$): the associated force of mortality (central death rate).
- Assume $\mu_{x,t}$ is constant within each integer age x and year t (i.e., $\mu_{x+r,t+s} = \mu_{x,t}$ for $r, s \in [0, 1)$), $\Rightarrow \mu_{x,t} = m_{x,t} = -\ln(p_{x,t})$.
- Let $[x_L, x_L] \times [T_1, T_2]$ be the study age-year window where mortality rates are available.
- Stand at the end of year t_U and want to fit the models with mortality data in an age-year window $[x_L, x_U] \times [t_L, t_U]$, where $T_1 \leq t_L$.
- Project mortality rates for the window $[x_L, x_U] \times [t_U + 1, T_2]$, and compare them with the observed ones in $[x_L, x_U] \times [t_U + 1, T_2]$.

The Lee-Carter Model (I)

$$\ln(m_{x,t}) = \alpha_x + \beta_x \cdot k_t + \epsilon_{x,t},\ x = x_L, \ldots, x_U,\ t = t_L, \ldots, t_U$$

- α_x and β_x are the age-specific constants;
- k_t is the time-varying index;
- for each x, error terms $\epsilon_{x,t} \overset{i.i.d}{\sim} N(0, \sigma_x^2)$, $t = t_L, \ldots, t_U$;
- two constraints: $\sum_t k_t = 0$ and $\sum_x \beta_x = 1$.
- the fitting year span $[t_L,\ t_U]$ and the fitting age span $[x_L,\ x_U]$ $(x_U - x_L + 1 = m)$.

Estimation of parameters:

- $\sum_{t=t_L}^{t_U} k_t = 0 \Rightarrow \hat{\alpha}_x = \frac{1}{n} \sum_{t=t_L}^{t_U} \ln(m_{x,t})$,
- $\sum_{x=x_L}^{x_U} \beta_x = 1 \Rightarrow \hat{k}_t = \sum_{x=x_L}^{x_U} [\ln(m_{x,t}) - \hat{\alpha}_x]$,
- Regressing $[\ln(m_{x,t}) - \hat{\alpha}_x]$ on $\hat{k}_t$ without the intercept term $\Rightarrow \hat{\beta}_x = \frac{\sum_t [\ln(m_{x,t}) - \hat{\alpha}_x] \times \hat{k}_t}{\sum_t \hat{k}_t^2}$,

The Lee-Carter Model (II)

- k_t is assumed to follow a random walk with drift process, $\hat{k}_t = \hat{k}_{t-1} + \theta^{LC} + \varepsilon_t$, where $\varepsilon_t \sim N(0, \sigma^2)$, or $\hat{k}_t - \hat{k}_{t-1} \sim N(\theta^{LC}, \sigma^2)$.
- The drift θ^{LC} is estimated by

$$\hat{\theta}^{LC} = \frac{\sum_{t=t_L+1}^{t_U} (\hat{k}_t - \hat{k}_{t-1})}{t_U - t_L} = \frac{\hat{k}_{t_U} - \hat{k}_{t_L}}{t_U - t_L} = \frac{\sum_{x=x_L}^{x_U} [\ln(m_{x,\,t_U}) - \ln(m_{x,\,t_L})]}{t_U - t_L}.$$

- $\hat{k}_{t_U+\tau} = \hat{k}_{t_U} + \sum_{t=1}^{\tau} (\hat{k}_{t_U+t} - \hat{k}_{t_U+t-1}) = \hat{k}_{t_U} + \hat{\theta}^{LC} \times \tau$
- The projected $\ln(m_{x,\,t_U+\tau})$ is obtained by a linear function of τ,

$$\ln(\hat{m}^{LC}_{x,\,t_U+\tau}) = \hat{\alpha}_x + \hat{\beta}_x \times \hat{k}_{t_U+\tau} = \ln(\hat{m}^{LC}_{x,\,t_U}) + \hat{\mu}^{LC}_x \times \tau,$$

with intercept (starting point) $\ln(\hat{m}^{LC}_{x,\,t_U}) = \hat{\alpha}_x + \hat{\beta}_x \times \hat{k}_{t_U}$ and slope (annual mortality decrement) $\hat{\mu}^{LC}_x = \hat{\beta}_x \cdot \hat{\theta}^{LC}$.

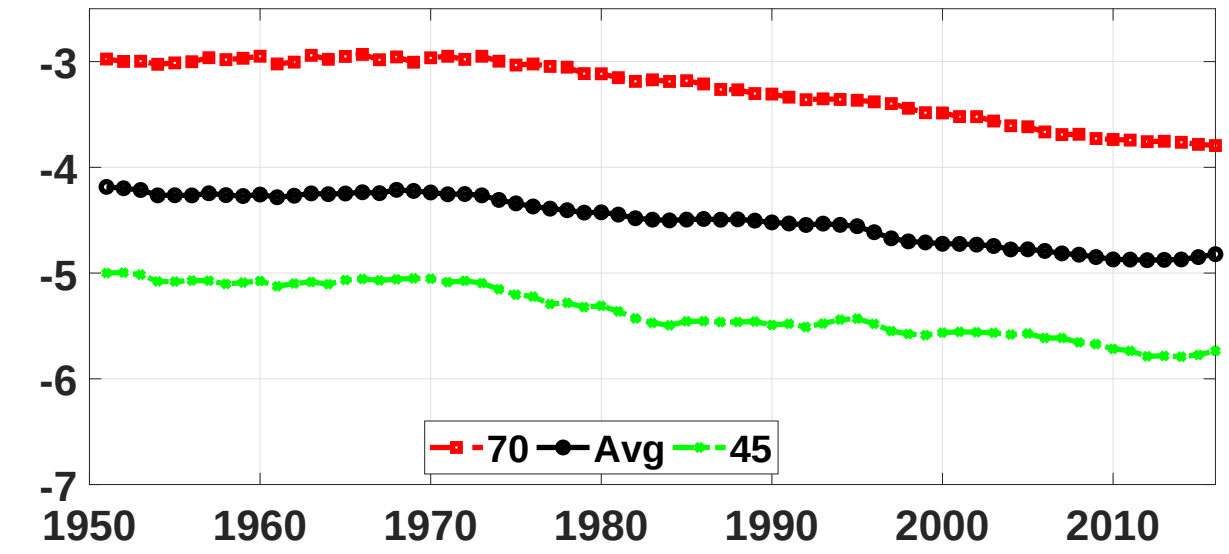
Random Walk with Drift Model

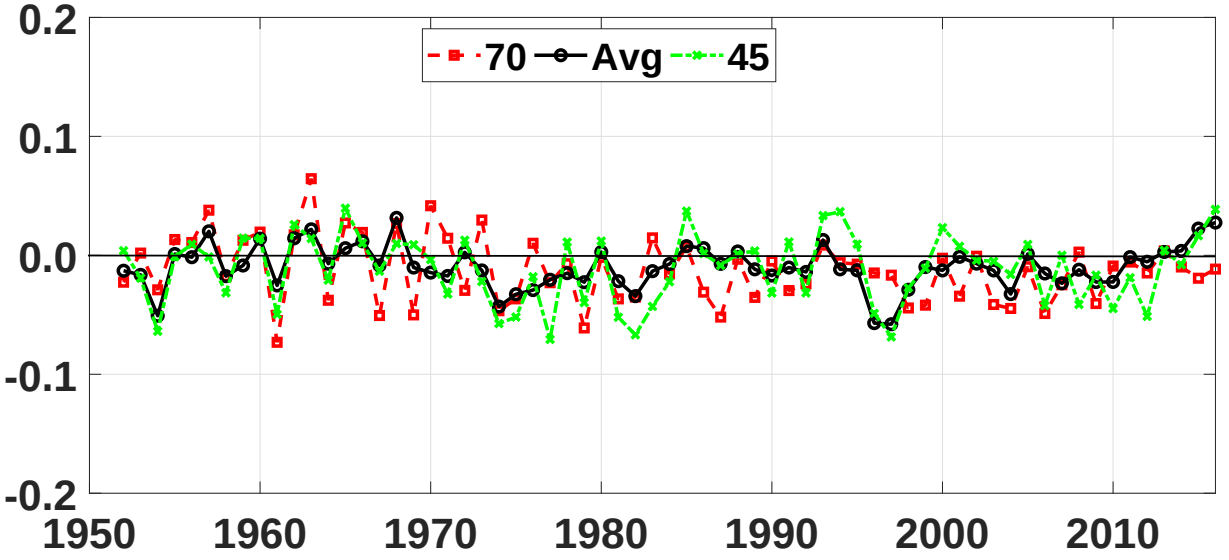
- $\ln(m_{x,t})$ is assumed to follow a random walk with drift process for each x, $\ln(m_{x,t+1}) = \ln(m_{x,t}) + \mu_x^{RW} + \varepsilon_{x,t}$, where $\varepsilon_{x,t} \overset{iid}{\sim} N(0, \sigma_x^2)$, or $Y_{x,t} \overset{\triangle}{=} \ln(m_{x,t_L+t}) - \ln(m_{x,t_L+t-1}) \overset{iid}{\sim} N(\mu_x^{RW}, \sigma_x^2)$, $t = 1, 2, \cdots$.
- The drift μ_x^{RW} is estimated by

$$\hat{\mu}_x^{RW} = \frac{\sum_{t=t_L}^{t_U-1} [\ln(m_{x,t+1}) - \ln(m_{x,t})]}{t_U - t_L} = \frac{\ln(m_{x,t_U}) - \ln(m_{x,t_L})}{t_U - t_L}.$$

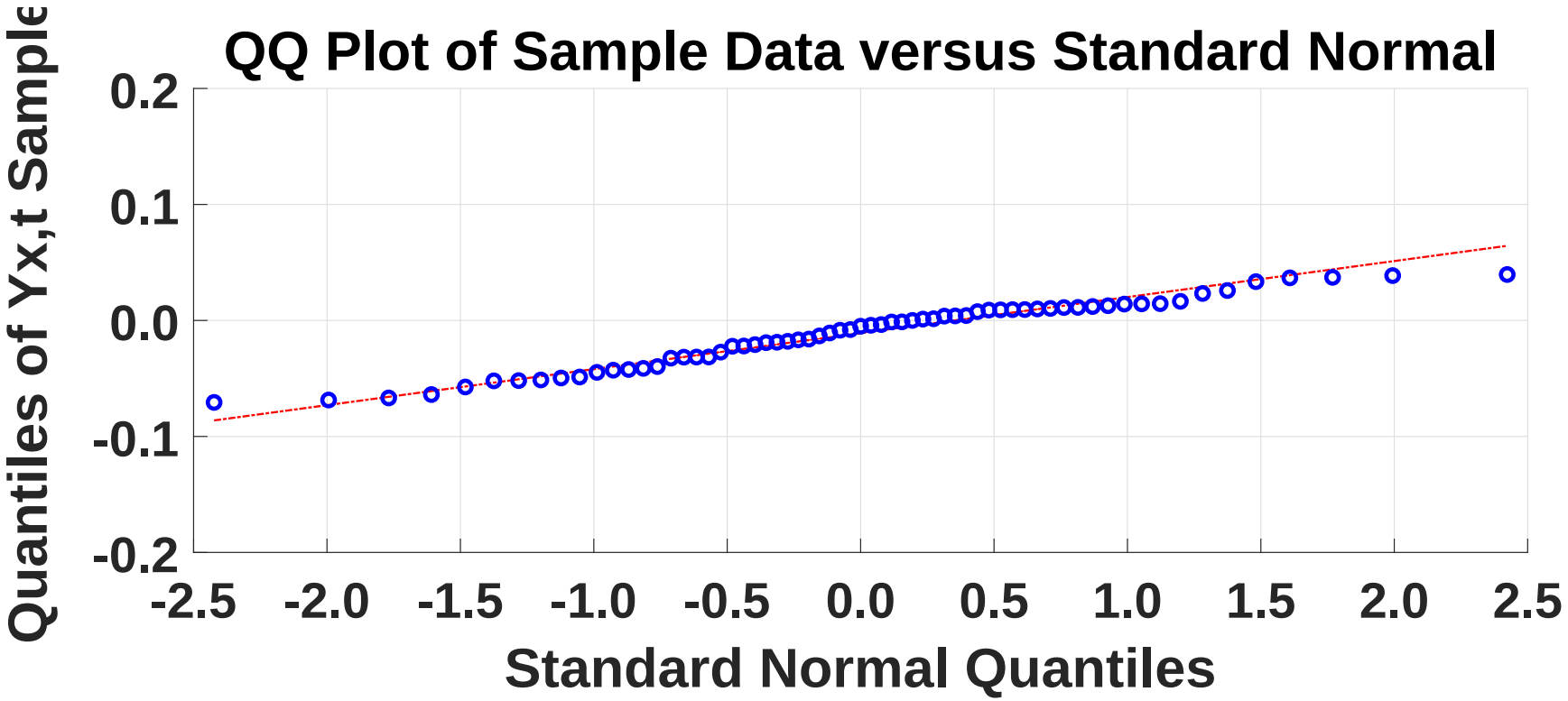
- The projected $\ln(m_{x,t_U+\tau})$ is obtained by a linear function of τ, $\ln(\hat{m}_{x,t_U+\tau}^{RW}) = \ln(m_{x,t_U}) + \sum_{t=1}^{\tau} [\ln(m_{x,t_U+t}) - \ln(m_{x,t_U+t-1})] = \ln(m_{x,t_U}) + \hat{\mu}_x^{RW} \times \tau$, with intercept $\ln(m_{x,t_U})$ and slope $\hat{\mu}_x^{RW}$.

$\ln(m_{x,t})$ and $Y_{x,t}$ for US males aged 45 and 70





Q-Q plot for US males aged 45



Bayesian Theory

- Assume we have observed $\tilde{Y} = \tilde{y}$, where $\tilde{Y} = (Y_1, \ldots, Y_n)^T$ and $\tilde{y} = (y_1, \ldots, y_n)^T$, and want to set a rate to cover Y_{n+1}.
- Let $Y_t|\Lambda, t = 1, \cdots, n+1$, be i.i.d random variables with probability density $f_{Y_t|\Lambda}(y|\lambda)$, where λ be a risk parameter from a r.v. Λ.
- the **prior** pdf of Λ: $\pi_\Lambda(\lambda)$.
- the **joint** pdf of $\tilde{Y}$ and Λ:
  $f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda) = f_{\tilde{Y}|\Lambda}(y_1, \cdots, y_n|\lambda) \cdot \pi_\Lambda(\lambda) \overset{ind.}{=} [\prod_{j=1}^n f_{Y_j|\Lambda}(y_j|\lambda)] \cdot \pi_\Lambda(\lambda)$
- the **posterior** pdf of Λ: the conditional pdf of Λ given $\tilde{Y} = \tilde{y}$, $\pi_{\Lambda|\tilde{Y}}(\lambda|\tilde{y}) = f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda)/f_{\tilde{Y}}(\tilde{y}) = [\prod_{j=1}^n f_{Y_j|\Lambda}(y_j|\lambda \cdot \pi_\Lambda(\lambda)]/f_{\tilde{Y}}(\tilde{y})$, where $f_{\tilde{Y}}(\tilde{y}) = \sum_\theta f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda)$.
- the **predictive** pdf: the conditional pdf of Y_{n+1} given $\tilde{Y} = \tilde{y}$ can be derived as $f_{Y_{n+1}|\tilde{Y}}(y_{n+1}|\tilde{y}) = \sum_\lambda f_{Y_{n+1}|\Lambda}(y_{n+1}|\lambda) \cdot \pi_{\Lambda|\tilde{Y}}(\lambda|\tilde{y})$.
- The mean of the predictive distribution (the **Bayesian premium**) is
  $$\begin{aligned} E[Y_{n+1}|\tilde{Y} = \tilde{y}] &= \sum_y y \cdot f_{Y_{n+1}|\tilde{Y}}(y|\tilde{y}) \\ &= \sum_\lambda \mu_{n+1}(\lambda) \cdot \pi_{\Lambda|\tilde{Y}}(\lambda|\tilde{y}) = E[\mu_{n+1}(\Lambda)|\tilde{Y} = \tilde{y}], \end{aligned}$$
  where $\mu_{n+1}(\lambda) = E[Y_{n+1}|\Lambda = \lambda] = \sum_y y \cdot f_{Y_{n+1}|\Lambda}(y|\lambda)$.

Bayesian Premium (an example)

Λ	$f_{Y_j\|\Lambda}(y_j = 0\|\lambda_i)$	$f_{Y_j\|\Lambda}(y_j = 1\|\lambda_i)$	$E[Y_j\|\Lambda = \lambda_i]$	$\pi_\Lambda(\lambda_i)$
λ_1	$1/2$	$1/2$	$1/2$	$1/2$
λ_2	$1/4$	$3/4$	$3/4$	$1/2$

$\tilde{y}$	$f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda_1)$	$f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda_2)$	$f_{\tilde{Y}}(\tilde{y})$	$\pi_{\Lambda\|\tilde{Y}}(\lambda_1\|\tilde{y})$	$\pi_{\Lambda\|\tilde{Y}}(\lambda_2\|\tilde{y})$
$(0,0)$	$4/32$	$1/32$	$5/32$	$4/5$	$1/5$
$(0,1)$	$4/32$	$3/32$	$7/32$	$4/7$	$3/7$
$(1,1)$	$4/32$	$9/32$	$13/32$	$4/13$	$9/13$

$\tilde{y}$	$E[Y_3\|\tilde{Y} = \tilde{y}] = \sum_{i=1}^{2} E[Y_3\|\Lambda = \lambda_i] \cdot \pi_{\Lambda\|\tilde{Y}}(\lambda_i\|\tilde{y})$
$(0,0)$	$1/2 \times 4/5 + 3/4 \times 1/5 = 11/20 = 0.55$
$(0,1)$	$1/2 \times 4/7 + 3/4 \times 3/7 = 17/28 = 0.61$
$(1,1)$	$1/2 \times 4/13 + 3/4 \times 9/13 = 35/52 = 0.67$

Bayesian Mortality Model

- $Y_t|(\Lambda = \lambda_z) \overset{iid}{\sim} N(\theta_z, \sigma_z^2), t = 1, \cdots$ with probability density at y_t,

$$f_{Y_t|\Lambda}(y_t|\lambda_z) = \frac{1}{\sqrt{2\pi} \cdot \sigma_z} \exp\left[-\frac{1}{2}\left(\frac{y_t - \theta_z}{\sigma_z}\right)^2\right] = \phi(y_t;\, \theta_z,\, \sigma_z^2).$$

- Risk parameter λ_z for age z corresponds to $(\theta_z,\, \sigma_z^2)$ which generates $\tilde{y}_z = (y_{z,\, t_L+1}, \cdots, z_{x,\, t_U})$, where $y_{z,\, t} \overset{\triangle}{=} \ln(m_{z,\, t_L+t}) - \ln(m_{z,\, t_L+t-1})$.
- $\hat{\theta}_z = \frac{\sum_{t_L+1}^{t_U} y_{z,\, t}}{t_U - t_L} = \frac{\sum_{t_L+1}^{t_U} [\ln(m_{z,\, t}) - \ln(m_{z,\, t-1})]}{t_U - t_L} = \frac{\ln(m_{z,\, t_U}) - \ln(m_{z,\, t_L})}{t_U - t_L} = \hat{\theta}_z^{RW}$.
- $\hat{\sigma}_z^2 = \frac{1}{t_U - t_L - 1} \sum_{t_L+1}^{t_U} [y_{z,\, t} - \hat{\theta}_z]^2 = (\hat{\sigma}_z^{RW})^2$.
- $\hat{\mu}_x^{BS} \overset{\triangle}{=} \hat{E}[Y_{n+1}|\tilde{Y} = \tilde{y}_x] = \sum_{z=x_L}^{x_U} \hat{\theta}_z\, \pi_{\Lambda|\tilde{Y}}(\lambda_z|\tilde{y}_x) = \sum_{z=x_L}^{x_U} \hat{\theta}_z^{RW} \frac{f_{\tilde{Y},\Lambda}(\tilde{y}_x,\, \lambda_z)}{f_{\tilde{Y}}(\tilde{y}_x)}$

$$= \frac{\sum_{z=x_L}^{x_U} \hat{\theta}_z^{RW} \cdot \pi_\Lambda(\lambda_z) \cdot \prod_{t=t_L+1}^{t_U} \phi(y_{x,\, t};\, \hat{\theta}_z,\, \hat{\sigma}_z^2)}{\sum_{w=x_L}^{x_U} \pi_\Lambda(\lambda_w) \cdot \prod_{t=t_L+1}^{t_U} \phi(y_{x,\, t};\, \hat{\theta}_w,\, \hat{\sigma}_w^2)}$$

  Given observations $\tilde{y}_x = (y_{x,\, t_L+1}, \cdots, y_{x,\, t_U})$, $\hat{\mu}_x^{BS}$ for age x is a weighted average of $\hat{\mu}_z^{RW}$s with the posterior density $\pi_{\Lambda|\tilde{Y}}(\lambda_z|\tilde{y}_x)$ as the weight.
- Assume $\pi_\Lambda(\lambda_z) = 1/m$, $z = x_L, \cdots, x_U$, where $m = x_U - x_L + 1$.

$\pi_{\Lambda|\tilde{Y}}(\lambda_z|\tilde{y}_x)$ vs z with fitting window $[25, 84] \times [1951, 2006]$

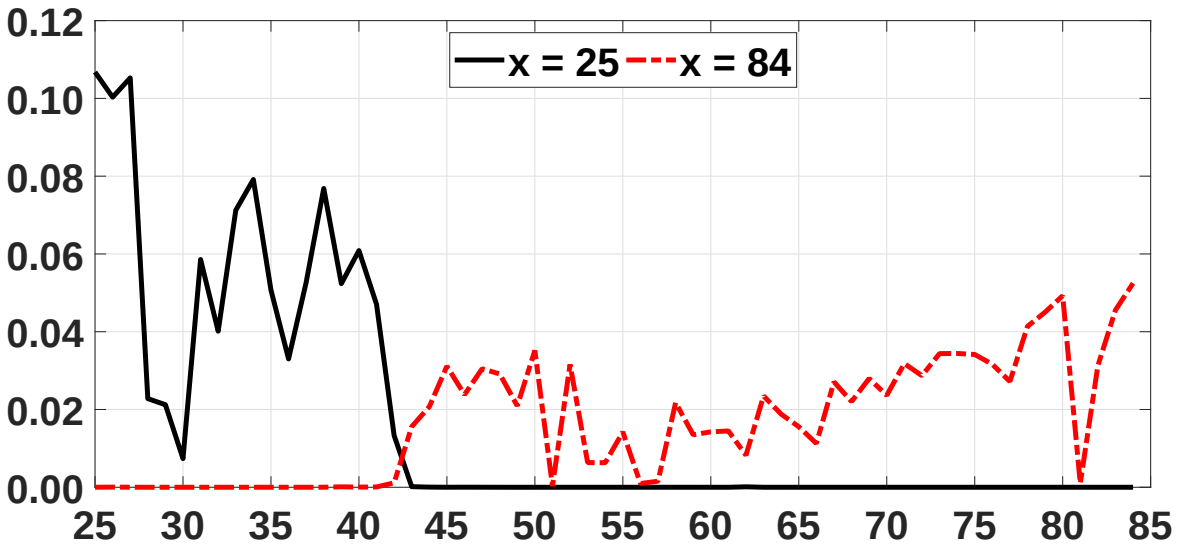
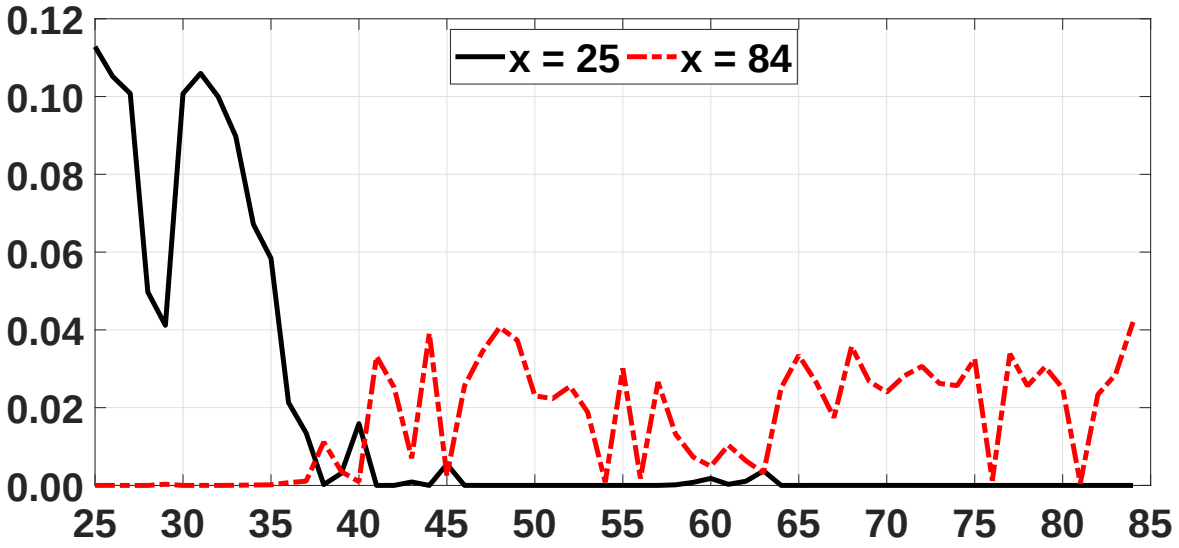


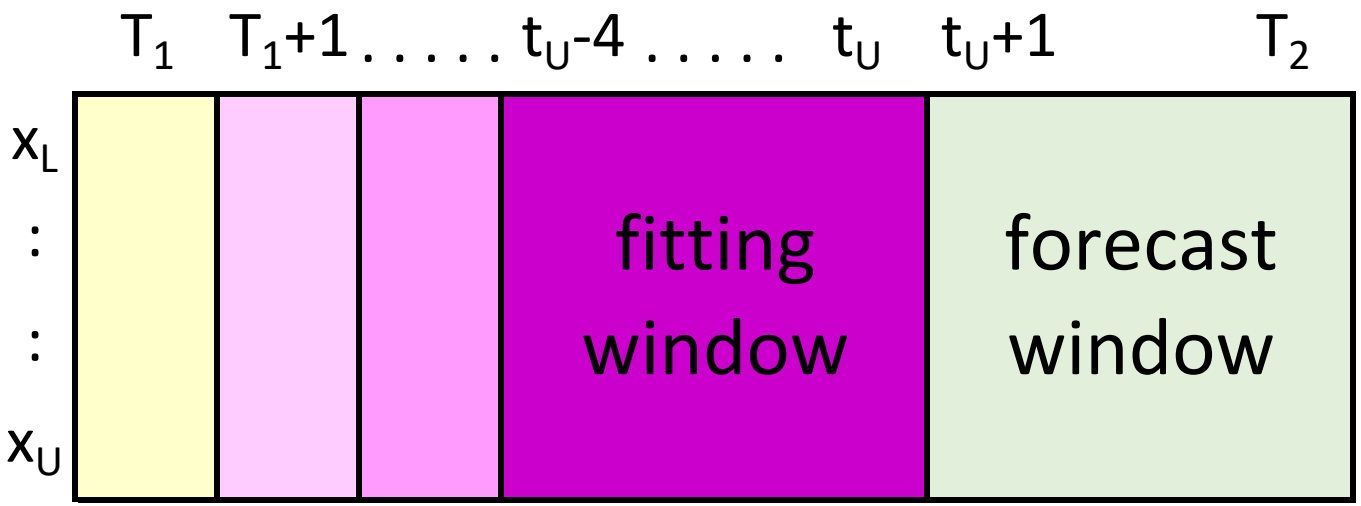
Figure: top: US males; bottom: US female



Models for comparisons

- LC-f model: $\ln(\hat{m}^{LC-f}_{x,\,t_U+\tau}) = \ln(\hat{m}^{LC}_{x,\,t_U}) + \hat{\mu}^{LC}_x \times \tau,\ \tau = 1, 2, \cdots,$ with intercept $\ln(\hat{m}^{LC}_{x,\,t_U})$ and slope $\hat{\mu}^{LC}_x = \hat{\beta}_x \cdot \hat{\theta}^{LC}$
- RW model: $\ln(\hat{m}^{RW}_{x,\,t_U+\tau}) = \ln(m_{x,\,t_U}) + \hat{\mu}^{RW}_x \times \tau,\ \tau = 1, 2, \cdots,$ with intercept $\ln(m_{x,\,t_U})$ and slope $\hat{\mu}^{RW}_x$.
- Bayesian model: $\ln(\hat{m}^{BS}_{x,\,t_U+\tau}) = \ln(m_{x,\,t_U}) + \hat{\mu}^{BS}_x \times \tau,\ \tau = 1, 2, \cdots,$ with intercept $\ln(m_{x,\,t_U})$ and slope $\hat{\mu}^{BS}_x = E[Y_{n+1}|\tilde{Y} = \tilde{y}_x]$.
- LC-r model: $\ln(\hat{m}^{LC-r}_{x,\,t_U+\tau}) = \ln(m_{x,\,t_U}) + \hat{\mu}^{LC}_x \times \tau,\ \tau = 1, 2, \cdots,$ with intercept $\ln(m_{x,\,t_U})$ and slope $\hat{\mu}^{LC}_x = \hat{\beta}_x \cdot \hat{\theta}^{LC}$
- Total intercepts over all ages for all models are preserved: $\sum_{x=x_L}^{x_U} \ln(\hat{m}^{LC}_{x,\,t_U}) = \sum_{x=x_L}^{x_U} \ln(m_{x,\,t_U})$.
- Total slopes over all ages for LC-f, LC-r and RW models are preserved: $\sum_{x=x_L}^{x_U} \hat{\mu}^{RW}_x = \sum_{x=x_L}^{x_U} \hat{\mu}^{LC} = \hat{\theta}^{LC}$.

Mortality Data



- study window: $[x_L,\ x_U] \times [T_1,\ T_2]$, where mortality data are available from the Human Mortality Database. We select $x_L = 25$, $x_U = 84$, $T_1 = 1951$, $T_2 = 2016$ for the study window $[25,\ 64] \times [1951,\ 2016]$.
- 3 forecasting windows: $[x_L,\ x_U] \times [t_U + 1,\ T_2]$, where $t_U = 2006, 1996, 1986$ for $[25,\ 84] \times [2007,\ 2016]$, $[25,\ 84] \times [1997,\ 2016]$ and $[25,\ 84] \times [1987,\ 2016]$.
- a series of fitting windows: $[x_L,\ x_U] \times [T_1,\ t_U]$, $[x_L,\ x_U] \times [T_1 + 1,\ t_U], \cdots, [x_L,\ x_U] \times [t_U - 4,\ t_U]$;
- study countries: both genders of the US, the UK, and Japan.

Forecasting Error Measures

- Data conversion using $q_{x,\,t} = 1 - e^{-m_{x,\,t}}$ under the assumption that $\mu_{x+r,\,t+s}$ is constant for $r, s \in [0,\,1)$.
- MAPE (Mean Absolute Percentage Error) over a forecasting window $[x_L,\,x_U] \times [t_U + 1,\,T_2]$ based on the fitting window $[x_L,\,x_U] \times [t_L,\,t_U]$:

$$MAPE^{[x_L,\,x_U]\times[t_L,\,t_U]}_{[x_L,\,x_U]\times[t_U+1,\,T_2]} = \frac{1}{T_2 - t_U} \cdot \frac{1}{m} \sum_{\tau=1}^{T_2-t_U} \sum_{x=x_L}^{x_U} \left| \frac{\hat{q}_{x,\,t_U+\tau} - q_{x,\,t_U+\tau}}{q_{x,\,t_U+\tau}} \right|.$$

- AMAPE (Average of $MAPE^{[x_L,\,x_U]\times[t_L,\,t_U]}_{[x_L,\,x_U]\times[t_U+1,\,T_2]}$ over all fitting windows $[x_L,\,x_U] \times [T_1,\,t_U], [x_L,\,x_U] \times [T_1 + 1,\,t_U], \cdots [x_L,\,x_U] \times [t_U - 4,\,t_U]$:

$$AMAPE_{[x_L,\,x_U]\times[t_U+1,\,T_2]} = \frac{1}{t_U - 4 - T_1 + 1} \sum_{t_L=T_1}^{t_U-4} MAPE^{[x_L,\,x_U]\times[t_L,\,t_U]}_{[x_L,\,x_U]\times[t_U+1,\,T_2]}.$$

- Comparison of the overall forecasting performances among mortality models is based on the $AMAPE_{[x_L,\,x_U]\times[t_U+1,\,T_2]}$ measure.

Average of $MAPE^{[25,84]\times[t_L,2006]}_{[25,84]\times[2007,2016]}$ over all 6 populations against $t_L = 1951, \cdots, 2002$

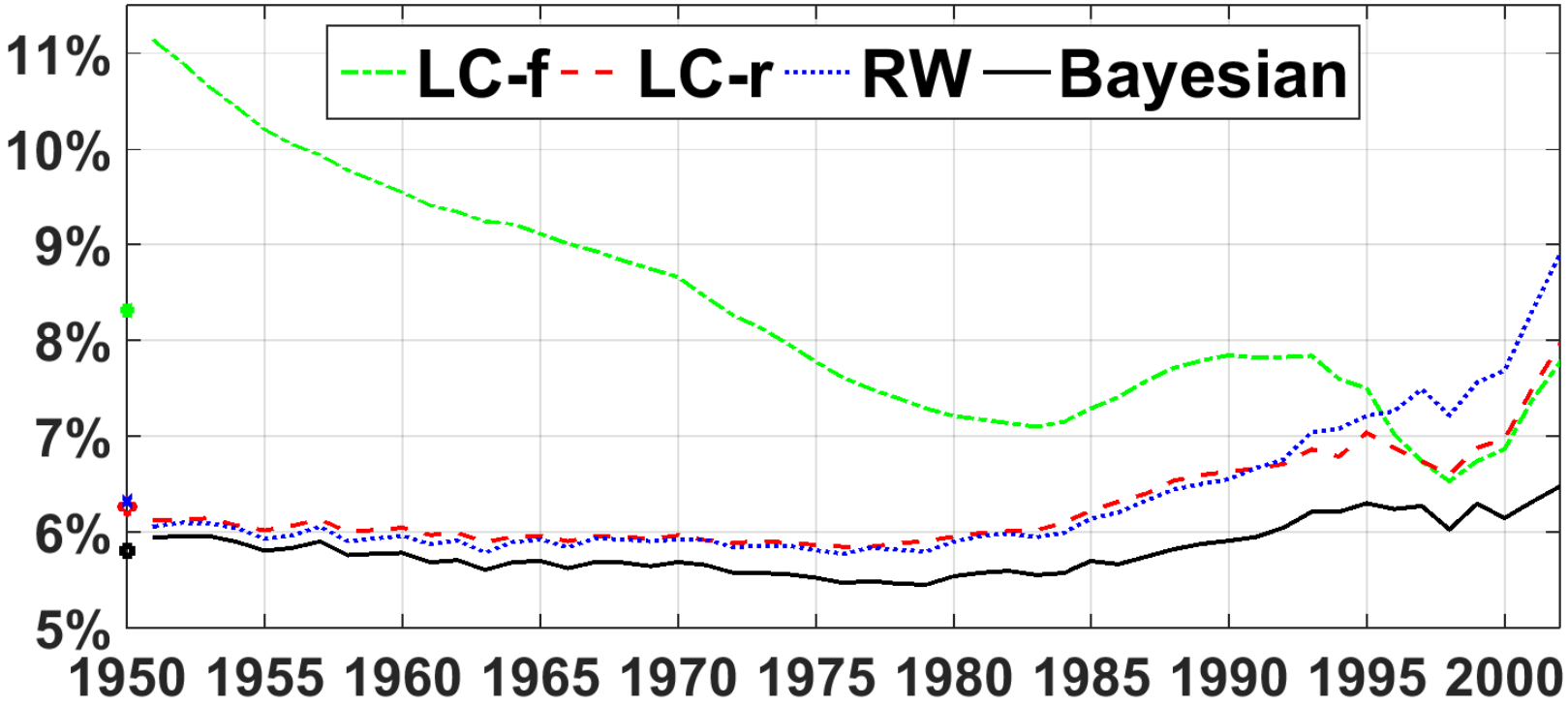


Figure: 10-year forecast for $[2007, 2016]$

Average of $MAPE^{[25,84]\times[t_L,1996]}_{[25,84]\times[1997,2016]}$ over all 6 populations against $t_L = 1951, \cdots, 1992$

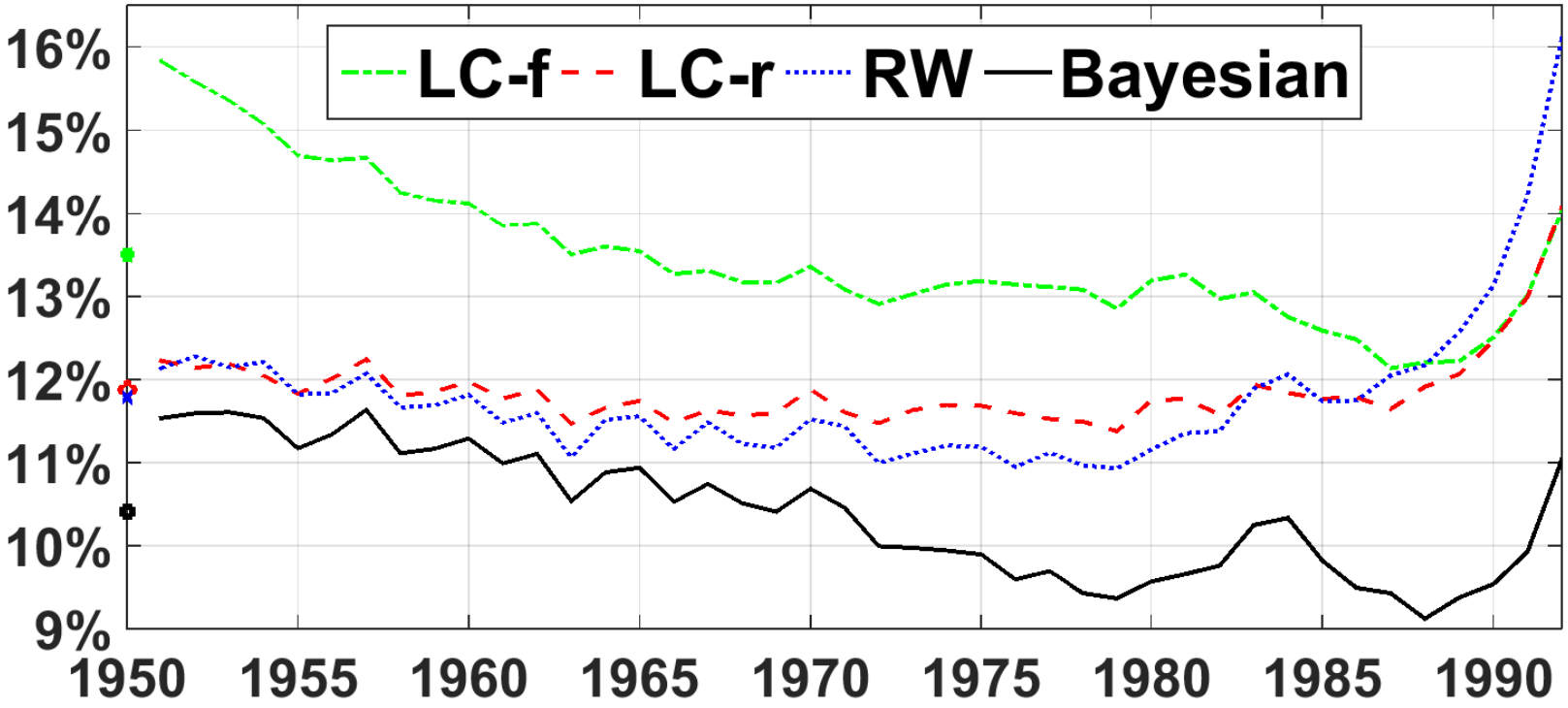


Figure: 20-year forecast for $[1997, 2016]$

Average of $MAPE_{[25,84]\times[1987,2016]}^{[25,84]\times[t_L,1986]}$ over all 6 populations against $t_L = 1951, \cdots, 1982$

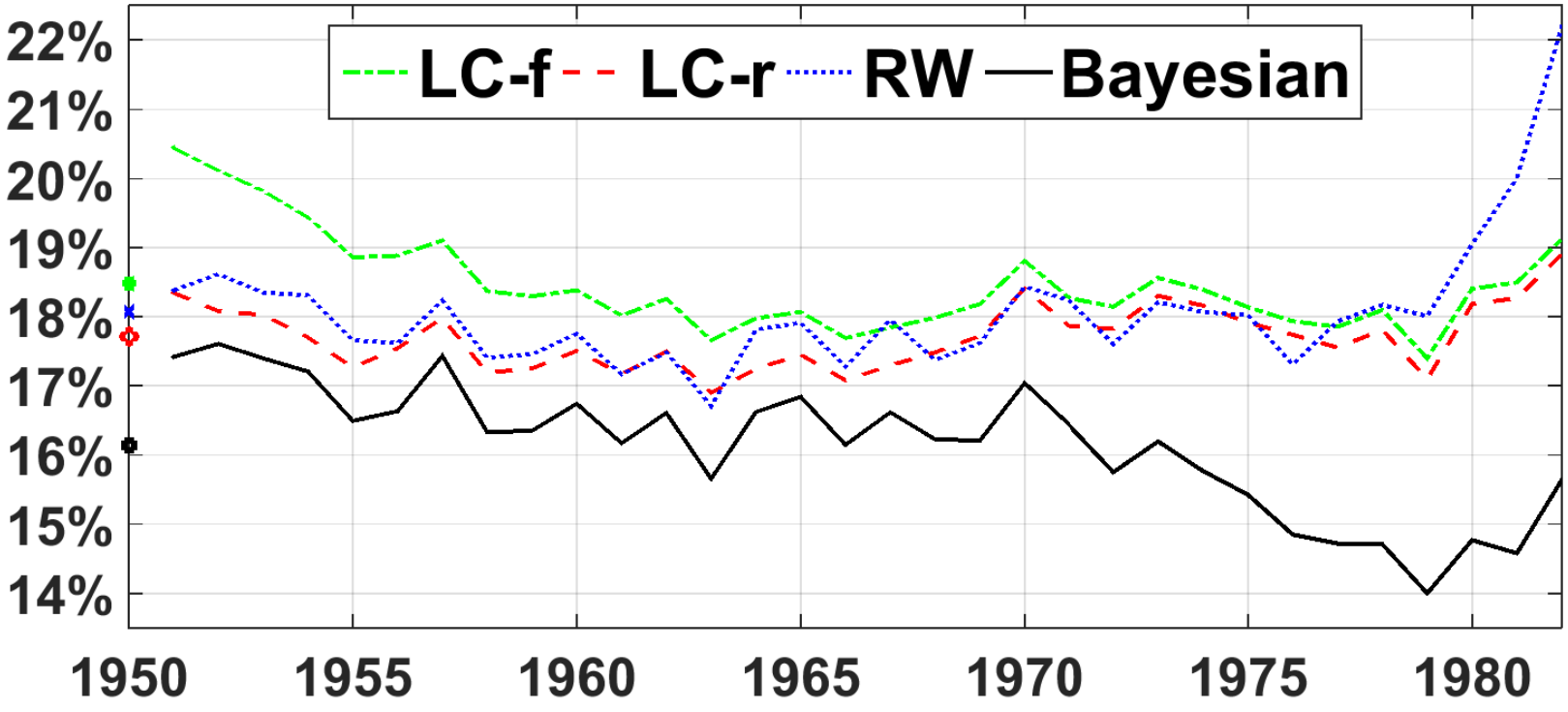


Figure: 30-year forecast for $[1987, 2016]$

SLOPE μ_x VERSUS x FOR US MALES

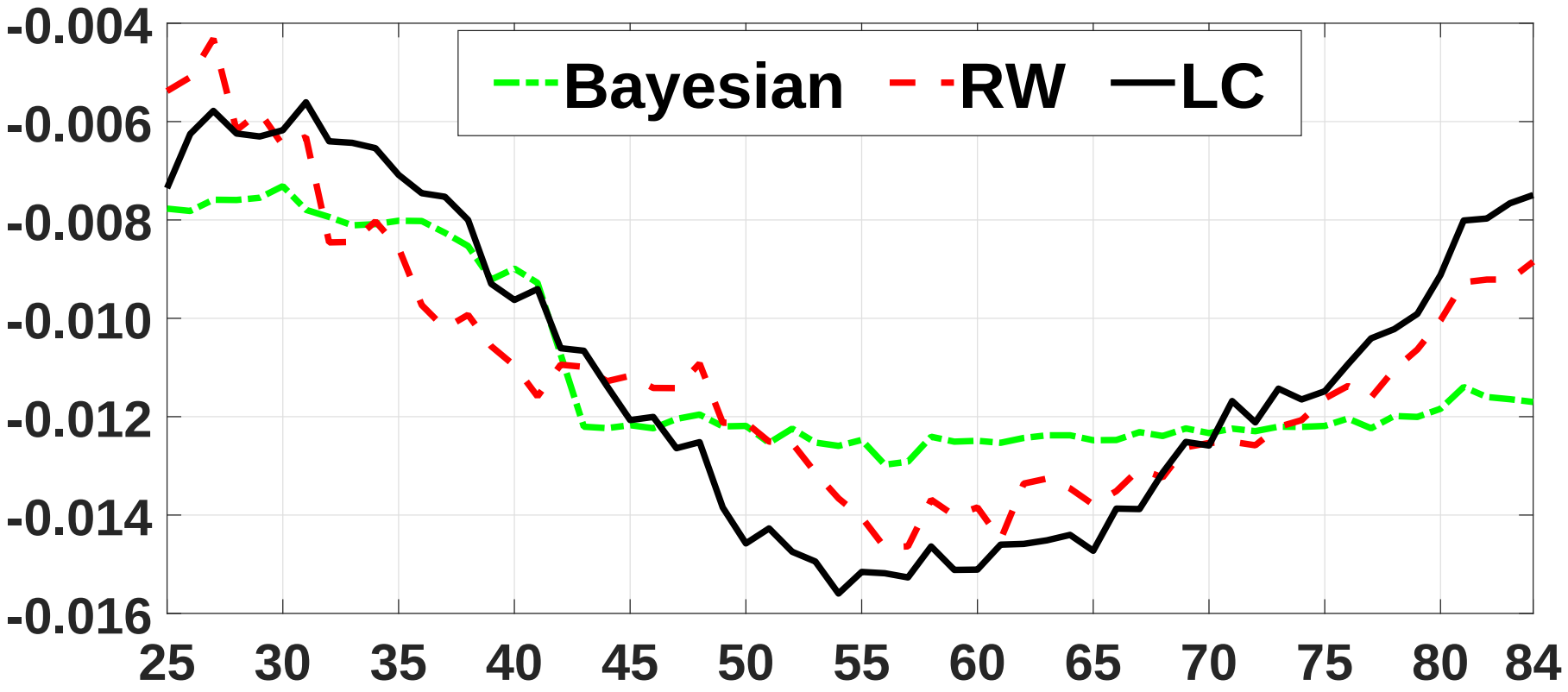


FIGURE: $MAPE^{[25,84]\times[1951,2006]}_{[25,84]\times[2007,2016]} = [4.50\%^{BS}, 4.81\%^{RW}, 8.60\%^{LC}]$

SLOPE μ_x VERSUS x FOR US MALES

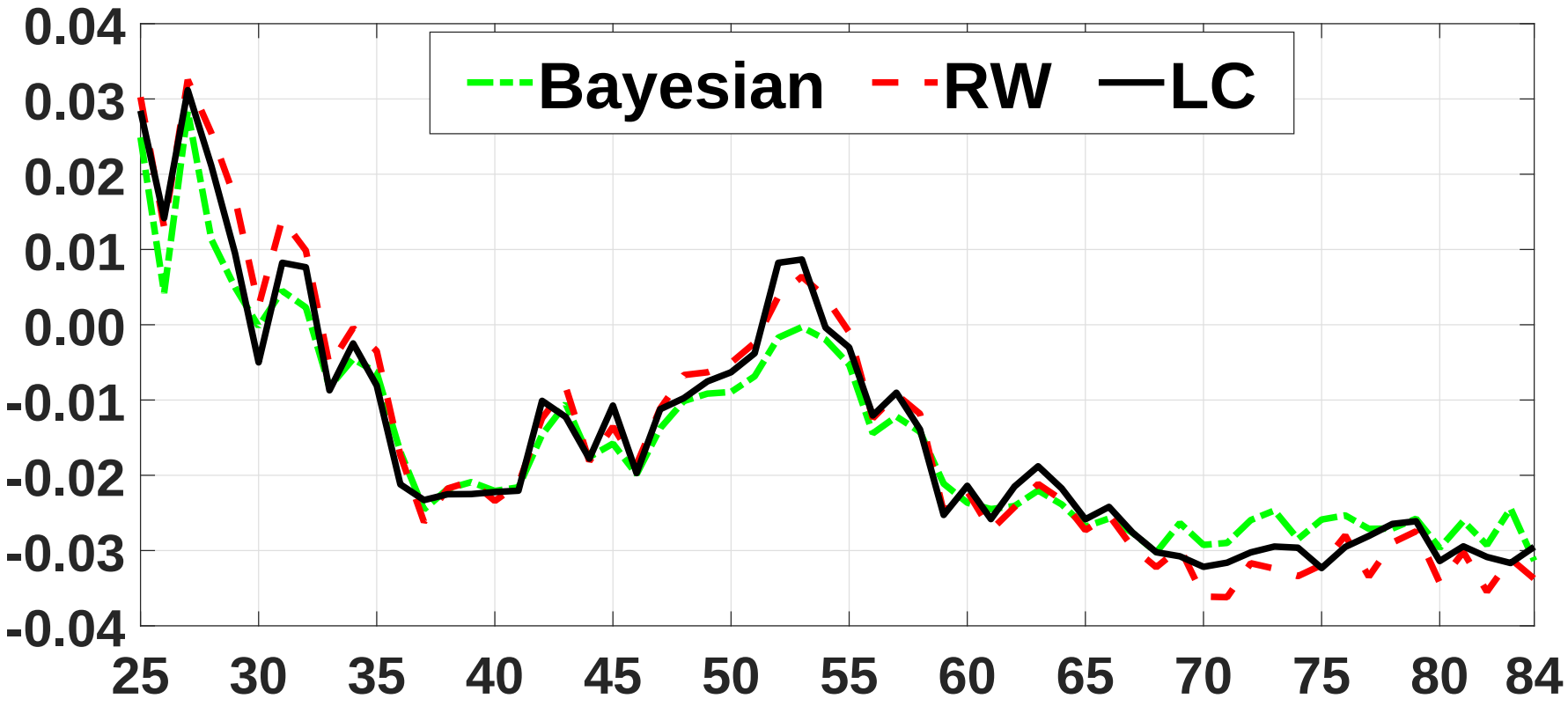


FIGURE: $MAPE^{[25,\,84]\times[2002,\,2006]}_{[25,\,84]\times[2007,\,2016]} = [6.96\%^{BS},\ 8.40\%^{RW},\ 7.60\%^{LC}]$

Forecasting Error Measures

$$AMAPE_{[25,\,84]\times[t_U+1,\,2016]}$$

Model	Avg-6	US-M	US-F	UK-M	UK-F	Japan-M	Japan-F
Panel A:AMAPE (%) for [25, 84]x[2007, 2016]							
LC-f	8.31%	8.55%	8.28%	9.63%	7.82%	7.27%	8.31%
LC-r	6.26%	5.79%	5.20%	7.05%	6.18%	5.91%	7.43%
RW	6.32%	5.69%	4.91%	7.23%	6.41%	6.00%	7.68%
BS	**5.81%**	**5.24%**	**4.58%**	**6.75%**	**5.82%**	**5.43%**	**7.03%**
Panel B: AMAPE (%) for [25, 84]x[1997, 2016]							
LC-f	13.50%	14.91%	9.18%	17.48%	12.69%	11.44%	15.28%
LC-r	11.87%	12.83%	8.34%	15.52%	10.61%	10.44%	13.49%
RW	11.78%	12.42%	8.33%	15.28%	10.78%	10.34%	13.54%
BS	**10.40%**	**10.92%**	**7.04%**	**13.57%**	**8.97%**	**9.07%**	**12.84%**
Panel C: AMAPE (%) for [25, 84]x[1987, 2016]							
LC-f	18.47%	14.66%	13.42%	23.71%	17.32%	18.58%	23.12%
LC-r	17.71%	13.95%	13.34%	22.44%	16.64%	17.94%	21.95%
RW	18.07%	15.71%	13.38%	22.50%	16.92%	17.86%	22.06%
BS	**16.14%**	**12.88%**	**11.64%**	**20.90%**	**13.56%**	**16.63%**	**21.24%**

Conclusions

- Change the starting location of forecast for the Lee-Carter model from the fitting value (LC-f) to the observed value (LC-r) can generally reduce the forecast error.
- The LC-r and RW models have similar forecasting performances based on the measure of Avg-6.
- The slope $\hat{\theta}_x^{RW}$ for age x under the RW model uses mortality data for a sole age x, and is not affected by $\hat{\theta}_z^{RW}$ for all $z \neq x$.
- Given observations $\tilde{Y} = \tilde{y}_x$, the slope $\hat{\theta}_x^{BS}$ for age x under the Bayesian model is a weighted average of $\hat{\mu}_z^{RW}$s (all slopes/drifts under the RW model) with the posterior density $\pi_{\Theta|\tilde{Y}}(\theta_z|\tilde{y}_x)$ as the weight:

  $\hat{\mu}_x^{BS} \triangleq E[Y_{n+1}|\tilde{Y} = \tilde{y}_x] = \sum_{z=x_L}^{x_U} \hat{\mu}_z^{RW} \cdot \pi_{\Theta|\tilde{Y}}(\theta_z|\tilde{y}_x)$
- Among the Bayesian, RW and LC-r models with the same starting location (intercept) for mortality forecast, the Bayesian model can better capture the downward mortality trend with more accurate slopes than the LC-r and RW models.

The End

Thank You