A Bayesian approach to modeling mortality rates

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Outline

1 Motivations

2 Mortality Models

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Motivations of developing effective mortality models

- Modeling the changes and dynamics of mortality rates is critical to the solvency of life insurers and social benefit programs.
- Mortality is also one of the key factors in pricing and reserving of life insurance and annuity products.
- Failing in capturing the downward trends in mortality rates would under-price/-reserve annuity products and then expose annuity providers and social security systems to the risk of financial insolvency.
Fitting window and forecasting window

\[ \begin{array}{c|c|c|c|c}
T_1 \hspace{1cm} & t_L \hspace{1cm} & t_U \hspace{1cm} & t_U+1 \hspace{1cm} & T_2 \\
X_L \hspace{1cm} & \text{fitting} \hspace{1cm} & \text{window} \hspace{1cm} & \text{forecasting} \hspace{1cm} & \text{window} \\
X_U \hspace{1cm} & \end{array} \]

- \( q_{x,t} \) (\( p_{x,t} = 1 - q_{x,t} \)): one-year death (survival) probability, and \( \mu_{x,t} \) (\( m_{x,t} \)): the associated force of mortality (central death rate).
- Assume \( \mu_{x,t} \) is constant within each integer age \( x \) and year \( t \) (i.e., \( \mu_{x+r,t+s} = \mu_{x,t} \) for \( r, s \in [0,1] \)), \( \Rightarrow \mu_{x,t} = m_{x,t} = -\ln(p_{x,t}) \).
- Let \([X_L, X_U] \times [T_1, T_2]\) be the study age-year window where mortality rates are available.
- Stand at the end of year \( t_U \) and want to fit the models with mortality data in an age-year window \([X_L, X_U] \times [t_L, t_U]\), where \( T_1 \leq t_L \).
- Project mortality rates for the window \([X_L, X_U] \times [t_U+1, T_2]\), and compare them with the observed ones in \([X_L, X_U] \times [t_U+1, T_2]\).
The Lee-Carter Model (I)

\[ \ln(m_x, t) = \alpha_x + \beta_x \cdot k_t + \epsilon_{x,t}, \quad x = x_L, \ldots, x_U, \quad t = t_L, \ldots, t_U \]

- \( \alpha_x \) and \( \beta_x \) are the age-specific constants;
- \( k_t \) is the time-varying index;
- for each \( x \), error terms \( \epsilon_{x,t} \) i.i.d. \( N(0, \sigma_x^2) \), \( t = t_L, \ldots, t_U \);
- two constraints: \( \sum_t k_t = 0 \) and \( \sum_x \beta_x = 1 \).
- the fitting year span \([t_L, t_U]\) and the fitting age span \([x_L, x_U]\) \((x_U - x_L + 1 = m)\).

Estimation of parameters:
- \( \sum_{t=t_L}^{t_U} k_t = 0 \Rightarrow \hat{\alpha}_x = \frac{1}{n} \sum_{t=t_L}^{t_U} \ln(m_x, t), \)
- \( \sum_{x=x_L}^{x_U} \beta_x = 1 \Rightarrow \hat{k}_t = \sum_{x=x_L}^{x_U} [\ln(m_x, t) - \hat{\alpha}_x], \)
- Regressing \([\ln(m_x, t) - \hat{\alpha}_x] \) on \( \hat{k}_t \) without the intercept term
  \( \Rightarrow \hat{\beta}_x = \frac{\sum_t [\ln(m_{x,t})-\hat{\alpha}_x] \times \hat{k}_t}{\sum_t \hat{k}_t^2}, \)
The Lee-Carter Model (II)

- $k_t$ is assumed to follow a random walk with drift process, 
  \[ \hat{k}_t = \hat{k}_{t-1} + \theta^{LC} + \varepsilon_t, \text{ where } \varepsilon_t \sim N(0, \sigma^2), \text{ or } \]
  \[ \hat{k}_t - \hat{k}_{t-1} \sim N(\theta^{LC}, \sigma^2). \]
- The drift $\theta^{LC}$ is estimated by
  \[ \hat{\theta}^{LC} = \frac{\sum_{t=t_L+1}^{t_U} (\hat{k}_t - \hat{k}_{t-1})}{t_U - t_L} = \frac{\hat{k}_{t_U} - \hat{k}_{t_L}}{t_U - t_L} = \frac{\sum_{x=x_L}^{x_U} [\ln(m_x, t_U) - \ln(m_x, t_L)]}{x_U - x_L}. \]
- \[ \hat{k}_{t_U + \tau} = \hat{k}_{t_U} + \sum_{t=1}^{\tau} (\hat{k}_{t_U + t} - \hat{k}_{t_U + t-1}) = \hat{k}_{t_U} + \hat{\theta}^{LC} \times \tau \]
- The projected $\ln(m_x, t_{U+\tau})$ is obtained by a linear function of $\tau$, 
  \[ \ln(\hat{m}_{x, t_{U+\tau}}^{LC}) = \hat{\alpha}_x + \hat{\beta}_x \times \hat{k}_{t_U + \tau} = \ln(\hat{m}_{x, t_U}^{LC}) + \hat{\mu}_x^{LC} \times \tau, \]
  with intercept (starting point) $\ln(\hat{m}_{x, t_U}^{LC}) = \hat{\alpha}_x + \hat{\beta}_x \times \hat{k}_{t_U}$ and slope (annual mortality decrement) $\hat{\mu}_x^{LC} = \hat{\beta}_x \cdot \hat{\theta}^{LC}$. 
Random Walk with Drift Model

- $\ln(m_x, t)$ is assumed to follow a random walk with drift process for each $x$, $\ln(m_x, t+1) = \ln(m_x, t) + \mu_x^{RW} + \varepsilon_x, t$, where $\varepsilon_x, t \sim iid \, N(0, \sigma_x^2)$, or $Y_x, t \triangleq \ln(m_x, t+1) - \ln(m_x, t) \sim iid \, N(\mu_x^{RW}, \sigma_x^2)$, $t = 1, 2, \cdots$.
- The drift $\mu_x^{RW}$ is estimated by
  $$
  \hat{\mu}_x^{RW} = \frac{\sum_{t=t_L}^{t_U-1} [\ln(m_x, t+1) - \ln(m_x, t)]}{t_U - t_L} = \frac{\ln(m_x, t_U) - \ln(m_x, t_L)}{t_U - t_L}.
  $$
- The projected $\ln(m_x, t_{U+\tau})$ is obtained by a linear function of $\tau$,
  $$
  \ln(\hat{m}_x^{RW, t_{U+\tau}}) = \ln(m_x, t_U) + \sum_{t=1}^{\tau} [\ln(m_x, t_{U+t}) - \ln(m_x, t_{U+t-1})] = \ln(m_x, t_U) + \hat{\mu}_x^{RW} \times \tau, \text{ with intercept } \ln(m_x, t_U) \text{ and slope } \hat{\mu}_x^{RW}.
  $$
\( \ln(m_{x,t}) \) AND \( Y_{x,t} \) FOR US MALES AGED 45 AND 70
QQ plot for US males aged 45
Bayesian Theory

- Assume we have observed $\tilde{Y} = \tilde{y}$, where $\tilde{Y} = (Y_1, \ldots, Y_n)^T$ and $\tilde{y} = (y_1, \ldots, y_n)^T$, and want to set a rate to cover $Y_{n+1}$.
- Let $Y_t|\Lambda, t = 1, \ldots, n + 1$, be i.i.d random variables with probability density $f_{Y_t|\Lambda}(y|\lambda)$, where $\lambda$ be a risk parameter from a r.v. $\Lambda$.
- the prior pdf of $\Lambda$: $\pi_{\Lambda}(\lambda)$.
- the joint pdf of $\tilde{Y}$ and $\Lambda$:
  \[ f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda) = f_{\tilde{Y}|\Lambda}(y_1, \ldots, y_n|\lambda) \cdot \pi_{\Lambda}(\lambda) \overset{ind.}{=} [\prod_{j=1}^n f_{Y_j|\Lambda}(y_j|\lambda)] \cdot \pi_{\Lambda}(\lambda) \]
- the posterior pdf of $\Lambda$: the conditional pdf of $\Lambda$ given $\tilde{Y} = \tilde{y}$, $\pi_{\Lambda|\tilde{Y}}(\lambda|\tilde{y}) = f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda)/f_{\tilde{Y}}(\tilde{y}) = [\prod_{j=1}^n f_{Y_j|\Lambda}(y_j|\lambda \cdot \pi_{\Lambda}(\lambda)]/f_{\tilde{Y}}(\tilde{y})$,
  where $f_{\tilde{Y}}(\tilde{y}) = \sum_\theta f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda)$.
- the predictive pdf: the conditional pdf of $Y_{n+1}$ given $\tilde{Y} = \tilde{y}$ can be derived as $f_{Y_{n+1}|\tilde{Y}}(y_{n+1}|\tilde{y}) = \sum_\lambda f_{Y_{n+1}|\Lambda}(y_{n+1}|\lambda) \cdot \pi_{\Lambda|\tilde{Y}}(\lambda|\tilde{y})$.
- The mean of the predictive distribution (the **Bayesian premium**) is $E[Y_{n+1}|\tilde{Y} = \tilde{y}] = \sum_y y \cdot f_{Y_{n+1}|\tilde{Y}}(y|\tilde{y})$
  \[ = \sum_\lambda \mu_{n+1}(\lambda) \cdot \pi_{\Lambda|\tilde{Y}}(\lambda|\tilde{y}) = E[\mu_{n+1}(\Lambda)|\tilde{Y} = \tilde{y}], \]
  where $\mu_{n+1}(\lambda) = E[Y_{n+1}|\Lambda = \lambda] = \sum_y y \cdot f_{Y_{n+1}|\Lambda}(y|\lambda)$. 
**Bayesian Premium (an example)**

| $\Lambda$ | $f_{Y_j|\Lambda}(y_j = 0|\lambda_i)$ | $f_{Y_j|\Lambda}(y_j = 1|\lambda_i)$ | $E[Y_j|\Lambda = \lambda_i]$ | $\pi_{\Lambda}(\lambda_i)$ |
|---|---|---|---|---|
| $\lambda_1$ | $1/2$ | $1/2$ | $1/2$ | $1/2$ |
| $\lambda_2$ | $1/4$ | $3/4$ | $3/4$ | $1/2$ |

| $\tilde{y}$ | $f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda_1)$ | $f_{\tilde{Y},\Lambda}(\tilde{y}, \lambda_2)$ | $f_{\tilde{Y}}(\tilde{y})$ | $\pi_{\Lambda|\tilde{Y}}(\lambda_1|\tilde{y})$ | $\pi_{\Lambda|\tilde{Y}}(\lambda_2|\tilde{y})$ |
|---|---|---|---|---|---|
| $(0, 0)$ | $4/32$ | $1/32$ | $5/32$ | $4/5$ | $1/5$ |
| $(0, 1)$ | $4/32$ | $3/32$ | $7/32$ | $4/7$ | $3/7$ |
| $(1, 1)$ | $4/32$ | $9/32$ | $13/32$ | $4/13$ | $9/13$ |

| $\tilde{y}$ | $E[Y_3|\tilde{Y} = \tilde{y}] = \sum_{i=1}^{2} E[Y_3|\Lambda = \lambda_i] \cdot \pi_{\Lambda|\tilde{Y}}(\lambda_i|\tilde{y})$ |
|---|---|
| $(0, 0)$ | $1/2 \times 4/5 + 3/4 \times 1/5 = 11/20 = 0.55$ |
| $(0, 1)$ | $1/2 \times 4/7 + 3/4 \times 3/7 = 17/28 = 0.61$ |
| $(1, 1)$ | $1/2 \times 4/13 + 3/4 \times 9/13 = 35/52 = 0.67$ |
Bayesian Mortality Model

- $Y_t|\Lambda = \lambda_z \sim \mathcal{N}(\theta_z, \sigma^2_z)$, $t = 1, \cdots$ with probability density at $y_t$,

$$f_{Y_t|\Lambda}(y_t|\lambda_z) = \frac{1}{\sqrt{2\pi} \cdot \sigma_z} \exp \left[ -\frac{1}{2} \left( \frac{y_t - \theta_z}{\sigma_z} \right)^2 \right] = \phi(y_t; \theta_z, \sigma_z).$$

- Risk parameter $\lambda_z$ for age $z$ corresponds to $(\theta_z, \sigma^2_z)$ which generates

$$\tilde{y}_z = (y_z, t_{L+1}, \cdots, z_x, t_U),$$

where $y_z, t \triangleq \ln(m_z, t_{L+t}) - \ln(m_z, t_{L+t-1}).$

$$\hat{\theta}_z = \frac{\sum_{t_{L+1}}^{t_U} y_z, t}{t_U-t_L} = \frac{\sum_{t_{L+1}}^{t_U} [\ln(m_z, t) - \ln(m_z, t-1)]}{t_U-t_L} = \frac{\ln(m_z, t_U) - \ln(m_z, t_L)}{t_U-t_L} = \hat{\theta}_R W.$$

$$\hat{\sigma}^2_z = \frac{1}{t_U-t_L-1} \sum_{t_{L+1}}^{t_U} [y_z, t - \hat{\theta}_z]^2 = (\hat{\sigma}^2_R W)^2.$$

$$\hat{\mu}_{BS}^x \triangleq \hat{\mathbb{E}}[Y_{n+1}|\tilde{Y} = \tilde{y}_x] = \sum_{z=x_L}^{x_U} \hat{\theta}_z \pi_{\Lambda|\tilde{y}}(\lambda_z|\tilde{y}_x) = \sum_{z=x_L}^{x_U} \hat{\theta}_R W \frac{f_{\tilde{Y}, \Lambda}(\tilde{y}_x, \lambda_z)}{f_{\tilde{Y}}(\tilde{y}_x)}$$

$$= \frac{\sum_{z=x_L}^{x_U} \hat{\theta}_R W \cdot \pi_{\Lambda}(\lambda_z) \cdot \prod_{t=t_{L+1}}^{t_U} \phi(y_x, t; \hat{\theta}_z, \hat{\sigma}^2_z)}{\sum_{w=x_L}^{x_U} \pi_{\Lambda}(\lambda_w) \cdot \prod_{t=t_{L+1}}^{t_U} \phi(y_x, t; \hat{\theta}_w, \hat{\sigma}^2_w)}$$

Given observations $\tilde{y}_x = (y_x, t_{L+1}, \cdots, y_x, t_U)$, $\hat{\mu}_{BS}^x$ for age $x$ is a weighted average of $\hat{\mu}_R W$'s with the posterior density $\pi_{\Lambda|\tilde{y}}(\lambda_z|\tilde{y}_x)$ as the weight.

- Assume $\pi_{\Lambda}(\lambda_z) = 1/m$, $z = x_L, \cdots, x_U$, where $m = x_U - x_L + 1$. 

\[ \]
$\pi_{\Lambda|\tilde{y}}(\lambda_z|\tilde{y}_x)$ vs $z$ with fitting window $[25, 84] \times [1951, 2006]$

**Figure:** top: US males; bottom: US female
Models for comparisons

- **LC-f model:** \( \ln(\hat{m}_{x, t_U + \tau}^{LC-f}) = \ln(\hat{m}_{x, t_U}^{LC}) + \hat{\mu}_{x}^{LC} \times \tau, \tau = 1, 2, \ldots \), with intercept \( \ln(\hat{m}_{x, t_U}^{LC}) \) and slope \( \hat{\mu}_{x}^{LC} = \hat{\beta}_{x} \cdot \hat{\theta}^{LC} \).

- **RW model:** \( \ln(\hat{m}_{x, t_U + \tau}^{RW}) = \ln(m_{x, t_U}) + \hat{\mu}_{x}^{RW} \times \tau, \tau = 1, 2, \ldots \), with intercept \( \ln(m_{x, t_U}) \) and slope \( \hat{\mu}_{x}^{RW} \).

- **Bayesian model:** \( \ln(\hat{m}_{x, t_U + \tau}^{BS}) = \ln(m_{x, t_U}) + \hat{\mu}_{x}^{BS} \times \tau, \tau = 1, 2, \ldots \), with intercept \( \ln(m_{x, t_U}) \) and slope \( \hat{\mu}_{x}^{BS} = E[Y_{n+1}|\tilde{Y} = \tilde{y}_{x}] \).

- **LC-r model:** \( \ln(\hat{m}_{x, t_U + \tau}^{LC-r}) = \ln(m_{x, t_U}) + \hat{\mu}_{x}^{LC} \times \tau, \tau = 1, 2, \ldots \), with intercept \( \ln(m_{x, t_U}) \) and slope \( \hat{\mu}_{x}^{LC} = \hat{\beta}_{x} \cdot \hat{\theta}^{LC} \).

Total intercepts over all ages for all models are preserved:
\[
\sum_{x=x_L}^{x_U} \ln(\hat{m}_{x}^{LC}) = \sum_{x=x_L}^{x_U} \ln(m_{x})
\]

Total slopes over all ages for LC-f, LC-r and RW models are preserved:
\[
\sum_{x=x_L}^{x_U} \hat{\mu}_{x}^{RW} = \sum_{x=x_L}^{x_U} \hat{\mu}_{x}^{LC} = \hat{\theta}^{LC}
\]
study window: $[x_L, x_U] \times [T_1, T_2]$, where mortality data are available from the Human Mortality Database. We select $x_L = 25$, $x_U = 84$, $T_1 = 1951$, $T_2 = 2016$ for the study window $[25, 64] \times [1951, 2016]$.


a series of fitting windows:
$[x_L, x_U] \times [T_1, t_U]$, $[x_L, x_U] \times [T_1 + 1, t_U]$, $\cdots$, $[x_L, x_U] \times [t_U - 4, t_U]$;

study countries: both genders of the US, the UK, and Japan.
Forecasting Error Measures

- Data conversion using $q_{x,t} = 1 - e^{-m_{x,t}}$ under the assumption that $\mu_{x+r,t+s}$ is constant for $r, s \in [0, 1)$.

- MAPE (Mean Absolute Percentage Error) over a forecasting window $[x_L, x_U] \times [t_U + 1, T_2]$ based on the fitting window $[x_L, x_U] \times [t_L, t_U]$:

$$\text{MAPE}_{[x_L, x_U] \times [t_L, t_U]}^{[x_L, x_U] \times [t_U+1, T_2]} = \frac{1}{T_2 - t_U} \cdot \frac{1}{m} \sum_{\tau=1}^{T_2-t_U} \sum_{x=x_L}^{x_U} \left| \frac{\hat{q}_{x,t_U+\tau} - q_{x,t_U+\tau}}{q_{x,t_U+\tau}} \right|.$$

- AMAPE (Average of MAPE) over all fitting windows $[x_L, x_U] \times [T_1, t_U], [x_L, x_U] \times [T_1 + 1, t_U], \ldots [x_L, x_U] \times [t_U - 4, t_U]$:

$$\text{AMAPE}_{[x_L, x_U] \times [t_U+1, T_2]} = \frac{1}{t_U - 4 - T_1 + 1} \sum_{t_L=T_1}^{t_U-4} \text{MAPE}_{[x_L, x_U] \times [t_L, t_U]}^{[x_L, x_U] \times [t_U+1, T_2]}.$$

- Comparison of the overall forecasting performances among mortality models is based on the $\text{AMAPE}_{[x_L, x_U] \times [t_U+1, T_2]}$ measure.
Average of $\text{MAPE}^{[25, 84] \times [t_L, 2006]}_{[25, 84] \times [2007, 2016]}$ over all 6 populations against $t_L = 1951, \ldots, 2002$

**Figure**: 10-year forecast for [2007, 2016]
Average of $\text{MAPE}^{[25, 84] \times [t_L, 1996]}_{[25, 84] \times [1997, 2016]}$ over all 6 populations against $t_L = 1951, \cdots, 1992$

**Figure:** 20-year forecast for [1997, 2016]
Average of $\text{MAPE}_{[25, 84] \times [t_L, 1986]}$ and $\text{MAPE}_{[25, 84] \times [1987, 2016]}$ over all 6 populations against $t_L = 1951, \ldots, 1982$

**Figure:** 30-year forecast for [1987, 2016]
Slope $\mu_x$ versus $x$ for US males

Figure: $\text{MAPE}^{[25, 84] \times [1951, 2006]} = [4.50\%^{BS}, 4.81\%^{RW}, 8.60\%^{LC}]$
Slope $\mu_x$ versus $x$ for US males

**Figure:** $\text{MAPE}^{[25, 84] \times [2002, 2006]} = [6.96\%^{\text{BS}}, 8.40\%^{\text{RW}}, 7.60\%^{\text{LC}}] \times [2007, 2016]$
### Forecasting Error Measures

**AMAPE**\([25, 84] \times [t_U + 1, 2016]\)

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<tbody>
<tr>
<td><strong>Panel A: AMAPE (%) for ([25, 84] \times [2007, 2016])</strong></td>
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<tr>
<td>LC-f</td>
<td>8.31%</td>
<td>8.55%</td>
<td>8.28%</td>
<td>9.63%</td>
<td>7.82%</td>
<td>7.27%</td>
<td>8.31%</td>
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<tr>
<td>LC-r</td>
<td>6.26%</td>
<td>5.79%</td>
<td>5.20%</td>
<td>7.05%</td>
<td>6.18%</td>
<td>5.91%</td>
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<tr>
<td>RW</td>
<td>6.32%</td>
<td>5.69%</td>
<td>4.91%</td>
<td>7.23%</td>
<td>6.41%</td>
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<tr>
<td>BS</td>
<td><strong>5.81%</strong></td>
<td><strong>5.24%</strong></td>
<td><strong>4.58%</strong></td>
<td><strong>6.75%</strong></td>
<td><strong>5.82%</strong></td>
<td><strong>5.43%</strong></td>
<td><strong>7.03%</strong></td>
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<tr>
<td><strong>Panel B: AMAPE (%) for ([25, 84] \times [1997, 2016])</strong></td>
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<tr>
<td>LC-f</td>
<td>13.50%</td>
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<td>9.18%</td>
<td>17.48%</td>
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<tr>
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<td>8.34%</td>
<td>15.52%</td>
<td>10.61%</td>
<td>10.44%</td>
<td>13.49%</td>
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<tr>
<td>RW</td>
<td>11.78%</td>
<td>12.42%</td>
<td>8.33%</td>
<td>15.28%</td>
<td>10.78%</td>
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<tr>
<td>BS</td>
<td><strong>10.40%</strong></td>
<td><strong>10.92%</strong></td>
<td><strong>7.04%</strong></td>
<td><strong>13.57%</strong></td>
<td><strong>8.97%</strong></td>
<td><strong>9.07%</strong></td>
<td><strong>12.84%</strong></td>
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<tr>
<td><strong>Panel C: AMAPE (%) for ([25, 84] \times [1987, 2016])</strong></td>
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<td>LC-f</td>
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<td>16.92%</td>
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<td>22.06%</td>
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<tr>
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<td><strong>12.88%</strong></td>
<td><strong>11.64%</strong></td>
<td><strong>20.90%</strong></td>
<td><strong>13.56%</strong></td>
<td><strong>16.63%</strong></td>
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Conclusions

- Change the starting location of forecast for the Lee-Carter model from the fitting value (LC-f) to the observed value (LC-r) can generally reduce the forecast error.
- The LC-r and RW models have similar forecasting performances based on the measure of Avg-6.
- The slope $\hat{\theta}_x^{RW}$ for age $x$ under the RW model uses mortality data for a sole age $x$, and is not affected by $\hat{\theta}_z^{RW}$ for all $z \neq x$.
- Given observations $\tilde{Y} = \tilde{y}_x$, the slope $\hat{\theta}_x^{BS}$ for age $x$ under the Bayesian model is a weighted average of $\hat{\mu}_z^{RW}$'s (all slopes/drifts under the RW model) with the posterior density $\pi_{\Theta | \tilde{Y}}(\theta_z | \tilde{y}_x)$ as the weight:
  \[
  \hat{\mu}_x^{BS} \triangleq E[Y_{n+1} | \tilde{Y} = \tilde{y}_x] = \sum_{z=x_L}^{x_U} \hat{\mu}_z^{RW} \cdot \pi_{\Theta | \tilde{Y}}(\theta_z | \tilde{y}_x)
  \]
- Among the Bayesian, RW and LC-r models with the same starting location (intercept) for mortality forecast, the Bayesian model can better capture the downward mortality trend with more accurate slopes than the LC-r and RW models.
The End

Thank You