

Historical Averages, Units Roots and Future Net Discount Rates: A Comprehensive Estimator

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I. Introduction

The forensic economics literature hosts a continuing debate about the appropriateness of using historical versus current rates to predict future net discount rates. If the net discount rate series is stationary, which means that shocks are transitory and the series reverts to a long-term mean value, estimates based on historical values are reasonable. Alternatively, if the series exhibits a unit root, then past observations have questionable predictive value and the best predictor of the next period's discount rate depends mainly on the current net discount rate.

Several analyses have been performed to examine the stationarity of net discount rates, and the literature is mixed in its conclusions. Whether the series is found to be stationary depends on the period and measures observed, as well as the type of statistical tests performed. The inherent uncertainty over the existence of a unit root does not help the practitioner choose between the long-term average and the current net discount rate as an appropriate estimator.

We derive an alternative estimator of future net discount rates based on the statistical properties of the underlying series. The estimator depends on both the length of the forecast horizon and the rate at which a time series converges to its equilibrium level in response to a shock (or its degree of persistence). For long forecast horizons, the estimator tends to resemble the long-term average. For short forecast horizons, the estimator more nearly resembles the current value. Similarly, if the degree of persistence is large (it converges slowly), the estimator tends to resemble the current rate whereas if persistence is low (it converges quickly), the estimator more nearly resembles the long-term average. In an important special case, where the net discount rate follows a first-order autoregressive process, the optimal estimator is a weighted average of the current net discount rate and the long-term mean of the net discount rate process.

The use of a data-based, weighted average optimal estimator has certain advantages. From a theoretical standpoint, the optimal estimator is more efficient than either the long-term average or the current net discount rate. Benchmark estimates suggest that the forecast error variance using the optimal estimator is less than half that of the long-term average or current net dis-

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count rate. Examples from historical U.S. data and from recent international data show that the optimal estimator would have performed better than the extreme alternatives.

Calculating and justifying the optimal weighting formula in the optimal estimator may be problematic in a forensic setting. Therefore, we also consider a compromise estimator that equally weights the current and long-term average net discount rates. The form of the optimal estimator and the time-series properties of net discount rates suggest that this compromise estimator is likely to be reasonably close to the efficient solution. The empirical examples show that the compromise estimator performs well, significantly outperforming both the long-term average and the current value. In fact, it matches the performance of the optimal estimator. This is good news for practitioners as it provides a simple, transparent method for predicting future net discount rates.

Section II defines the issue of mean revision versus a unit root more fully and contains a review of existing literature regarding the time series nature of net discount rates. In Section III, we derive the optimal estimator. Section IV presents a detailed analysis of the efficiency of this estimator for the special case where the net discount rate follows a first order autoregressive process. Section V demonstrates how the optimal and compromise estimators work in practice for historical U.S. data and recent international data. It is followed by a conclusion in Section VI.

II. Literature Review

This section begins with a general model reflecting alternative time series hypotheses regarding net discount rates. The expected parameter results that would support either stationarity of the net discount rate or a unit root are discussed. Then the literature is reviewed to show conflicting results.

A basic time series model of the net discount rate can be described as:

$$(1) \quad ndr_t = \beta_0 + \beta_1 ndr_{t-1} + \beta_2 D_t^{\bar{}} + \beta_3 DD_t^{\bar{}} + \beta_4 t + \beta_5 DD_t^{\bar{}} t + \sum_{i=1}^n \theta_i \Delta ndr_{t-i} + \varepsilon_t$$

where ndr_t is the net discount rate in period t , $D_t^{\bar{}}$ is a dummy variable equal to 1 in year t and zero in all other years, $DD_t^{\bar{}}$ is a second dummy variable equal to 1 in the years t through T and zero before t .¹

If $\beta_1 = 1$ the process exhibits a unit root. A shock to the process is permanent and the best predictor of the next period's discount rate is the current discount rate. If $\beta_1 \neq 1$, the process is stationary. If the other coefficients are zero, the net discount rate follows an *iid* process with mean β_0 and variance σ_ε^2 . If $\beta_4 \neq 0$, the process is trending either upward or downward as indicated by the sign of β_4 and is trend stationary.

In equation (1), three special cases are covered by the dummy variables $D_t^{\bar{}}$ and $DD_t^{\bar{}}$. The first dummy variable reflects a one-time break in the data in

¹This is the standard Perron (1997) equation used to test for unit roots augmented with a break in the intercept, slope or both.

period t . The second reflects movement to a new mean value starting in period t and continuing thereafter. The combination of DD_t^t and t allows the trend to shift starting in period t . If $\beta_2 \neq 0$, and the process is stationary, it will return to its original mean value. If $\beta_3 \neq 0$, and the process is stationary, it will revert to a new mean value. If $\beta_5 \neq 0$, the net discount rate will have a different trend in the years t through T than it did in the years zero through t .

A variety of studies have examined the stationarity of net discount rates. The studies have used a variety of time periods ranging from the 1950s to 2000. Observations were either by month, quarter or year. A few studies used real net discount rates while most used rates not adjusted for inflation. All but one study used interest rates on U.S. Treasury securities; the particular securities ranged across 91-days, 6-months, 1-year, 3-years and 10-years.² Wage rates included all private non-agricultural workers, production workers or disaggregated sectors of the labor market. Some studies used average hourly wages. Others used average weekly earnings. A variety of statistical techniques were used to test for unit roots or breaks in the series.

An early study that found stationarity of the net discount rate was Haslag, et al. (1991). They tested for stationarity of inflation adjusted interest rates and wages and found each series to be non-stationary. However, when they used the two measures to calculate a real net discount rate, they found that series to be stationary. In a companion piece, Haslag, et al. (1994), using two separate statistical tests, again rejected the null hypothesis of a unit root. They added more observations to the sample and based on their updated statistical work concluded that "significant nonlinearities are not present in the net discount ratio and that the variable can be represented as a stationary time series." (p. 517)

Gamber and Sorensen (1993) found that when they imposed a shift in the mean of the net discount rate series in 1979, they were able to reject the null hypothesis of a unit root and concluded that "the optimal forecast for the net discount rate is its mean value since the last mean shift." (p. 77) In a second paper, Gamber and Sorensen (1994) initially found that the net discount rate series was not stationary once serial correlation was eliminated from the residuals. However, they found that "the source of the nonstationarity appears to be a one-time shift in the mean of the series." (p. 505) Hays, et al. (2000) extended Gamber and Sorensen's (1994) work. They found that net discount rates were mean reverting in the long run, but could be persistent in the short run.

Johnson and Gelles (1996) also found differences in means of the net discount rates across pre-1966 and post-1980 periods, but provided little statistical analysis. Horvath and Sattler (1997), extending Johnson and Gelles, found statistical support for a regime change in 1980. Payne, et al. (1999) also suggested that a regime change occurred in the yield on U.S. Treasury securities in 1980. Using Person's unit root test that allows for one pre-specified structural break in the series, they rejected the null hypothesis of a unit root and concluded that "the variability of the net discount rate is largely due to transi-

²The outlier used high grade municipal bonds.

tory components. There is a tendency for the respective net discount rates to revert back to their long-run mean levels." (p. 222)

Sen, et al. (2000) eschewed the assumption of a change in the series in 1980. Instead, they tested for a unit root in a net discount rate series against the trend-break stationarity alternative with the break occurring at an unknown date. Sen, et al. found that "the net discount rate is best characterized as a stationary process with a single shift in the mean and that this shift occurred during 1978:3." (p. 29) In a companion piece that covered a longer time series, Sen, et al. (2002) found a structural shift in the first quarter of 1981 and concluded that "the practitioner should use the average net discount rate over the post-break period (1982:2-2000:3) rather than the average over the entire period." (p. 97)

In one of the latest additions to the literature, Braun, et al. (2005) argued that standard methods of looking for units roots may err on the side of not finding a unit root when there are breaks in the time series. They examined the stationarity of net discount rates using the "two-break minimum LM (Lagrange Multiplied) unit root test developed by Lee and Strazicich (2003)." (pp. 470-71) Using their test, Braun, et al. found two breaks in the net discount rate series. The timing of the breaks depended on the financial instrument used for the interest rate. Further, using their methodology, they could not reject the null hypothesis of a unit root. Hence, they argued the net discount rate series was not mean reverting.

The literature has also looked at medical net discount rates. As in the previously discussed literature, whether medical net discount rates are stationary depends on the sample and statistical tests used in the analysis.³

Based on the compendium of this research, it is unclear whether a net discount rate estimator that is derived from a stationary process is any better than one that is not. Therefore, a better predictor of the rate may be one that accounts for both historical and unit root elements of the series. Such an estimator is proposed in the next section. It is a weighted average of unit root and time series properties of the discount rate.

III. The Model

The literature proposes two extremes for estimating future discount rates from their past. The random walk estimator uses only the current rate and the long-term average uses an average of all past rates. These two extreme estimators correspond to two extreme assumptions on the data: either a pure random walk or a pure white noise process. We propose a third estimator that is the optimal linear predictor, given current and past observations. In a special, but empirically relevant case, the optimal estimator turns out to be a simple weighted average of the two extreme estimators.

We first must be precise about what we mean by "future net discount rates." In forensic economic settings, we typically require a single value that summarizes the future course of net discount rates. The number is presumably some average of net discount rates thought to prevail over the relevant forecast

³See, for example, Ewing, et al. (2001, 2003).

horizon. We choose to define the Future Net Discount Rate, $FNDR$, as a geometrically declining weighted average of future net discount rates over a horizon of n periods,

$$(2) \quad FNDR_{t,n} = \frac{1-\gamma}{1-\gamma^n} \sum_{j=0}^{n-1} \gamma^j (ndr_{t+1+j}).$$

In equation (2), $FNDR_{t,n}$ is the future net discount rate in period t projected out n years and ndr_{t+1+j} is the net discount rate in period $t+1+j$. The parameter γ governs the decay in the weighting scheme. For γ close to unity, we have a simple equally weighted average. For γ less than one, closer observations receive more weight than distant observations. By analogy to the term structure of interest rates literature, Shiller (1979), we suggest choosing $\gamma = 1/(1 + \bar{R})$ where \bar{R} is the long-run mean of the net discount rate process. In any case, our results are relatively insensitive to the choice of γ in the neighborhood of unity.

Although equation (2) is a convenient definition of the $FNDR$, optimal estimators of $FNDR$ turn out to be complicated and messy for all but the simplest time-series processes on ndr_t . We propose, therefore, to approximate (2) with what we define as the Permanent Net Discount Rate, $PNDR$, an *infinite* distributed lag on future net discount rates,

$$(3) \quad PNDR_t = (1-\lambda) \left(\sum_{j=0}^{\infty} \lambda^j (ndr_{t+1+j}) \right)$$

Equation (3) expresses the permanent rate as a geometrically declining weighted average of all future short-term rates. Large values of λ correspond to averages of future net discount rates over a very long horizon whereas small values correspond to averages over shorter horizons. We provide a procedure for selecting λ in a later part of this section.

In keeping with the net discount rate literature, we derive an estimator of $PNDR_t$ that depends only on past values of the net discount rate. We first impose some restrictions on the net discount rate process. Let ndr_t have an autoregressive representation:

$$(4) \quad B(L)ndr_t = a + e_t,$$

where $B(L) = 1 - \beta_1 L - \beta_2 L^2 - \dots$, is a polynomial in the lag operator, L , (defined as $L^k ndr_t = ndr_{t-k}$) and e_t is a fundamental white noise process with a finite mean and variance. Also assume that ndr_t is of exponential order less than λ^{-1} . This allows for a possible unit root in the net discount rate process, but rules out explosive growth.

Following the methodology outlined in Sargent (1987),

$$E_t(PNDR_t | ndr_t, ndr_{t-1}, \dots) = \widehat{PNDR}_t$$

can be expressed as

$$(5) \quad \widehat{PNDR}_t = \frac{(1-\lambda)(B(\lambda) - B(L))L^{-1}}{B(\lambda)(1-\lambda L^{-1})} ndr_t + \frac{\alpha}{B(\lambda)}.$$

Alternatively, using summation notation, equation (5) can be expressed as

$$(6) \quad \widehat{PNDR}_t = \frac{(1-\lambda) \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} ndr_{t-k} \beta_{j+1} \lambda^{j-k}}{1 - \sum_{i=1}^{\infty} \beta_i \lambda^i} + \frac{\alpha}{1 - \sum_{i=1}^{\infty} \beta_i \lambda^i}.$$

The second term in either equation (5) or (6) will depend, through α , on the deterministic components of the process—the mean and trend.⁴ In the case of a finite order autoregressive process of order k , the first term in (5) or (6) can be seen to involve only the current and $k-1$ lagged values of the net discount rate process.

To make the estimator \widehat{PNDR}_t operational, we must have a procedure for selecting λ . We wish to choose λ in a manner to make \widehat{PNDR}_t a good approximation to $\widehat{FNDR}_{t,n}$, the estimator of the true objective function, shown in equation (2). There is one special case in which the approximation can be made exact. When the underlying process on net discount rates is a first order autoregressive process, $ndr_t = \alpha + \rho(ndr_{t-1}) + e_t$, \widehat{PNDR}_t and $\widehat{FNDR}_{t,n}$ will be of the same form. In particular, the estimate of the true, truncated objective function can be shown to be,⁵

$$(7) \quad \widehat{FNDR}_{t,n} = \frac{\rho(1-\gamma)(1-(\rho\gamma)^n)}{(1-\rho\gamma)(1-\gamma^n)} ndr_t + \left(\frac{1}{(1-\rho)} - \frac{\rho(1-\gamma)(1-(\rho\gamma)^n)}{(1-\rho)(1-\rho\gamma)(1-\gamma^n)} \right) \alpha,$$

and the estimator for the infinite horizon version, (3), can be shown to be

$$(8) \quad \widehat{PNDR}_t = \frac{\rho(1-\lambda)}{1-\rho\lambda} ndr_t + \frac{1}{1-\rho\lambda} \alpha.$$

Equating coefficients on ndr_t in equations (7) and (8) implicitly defines λ as a function of ρ , γ and n :

$$(9) \quad \lambda = \frac{(1-\Omega)}{(1-\gamma\Omega)},$$

Where

⁴Note that if the net discount rate process is stationary, we have, $\alpha = \mu(1 - \sum_{i=1}^{\infty} \beta_i)$, where μ is the (unconditional) mean of the net discount rate process.

⁵Note that equations (7) and (8) are derived in Appendix 2.

$$(10) \quad \Omega = \frac{(1-\gamma)(1-(\gamma\rho)^n)}{(1-\gamma\rho)(1-\gamma^n)}.$$

Our strategy is to perform a preliminary regression that approximates the net discount rate as a first-order autoregressive process. We then use the estimate of ρ from the preliminary regression to estimate λ . If the process used in the forecast is also first-order autoregressive, the procedure is exact. If the time-series process used to perform the forecast is of higher order, then using equation (5) with λ estimated from equation (10) is likely to provide a good approximation. Appendix 1 provides tables that give solutions for λ as functions of n , ρ and γ from equation (9).

IV. A Special Case

To better understand the optimal estimator of the previous section and to provide some evidence of its relative superiority as compared to the random walk or long-term estimators, it is useful to examine an important special case. Suppose ndr_t follows a stationary first-order autoregressive process with AR parameter, ρ :

$$(11) \quad ndr_t = \alpha + \rho ndr_{t-1} + e_t.$$

Here, equations (6) and (7) collapse to

$$(12) \quad \widehat{PNDR}_t = \frac{\rho(1-\lambda)}{1-\rho\lambda} ndr_t + \frac{1-\rho}{1-\rho\lambda} \mu,$$

where $\mu = \alpha/(1-\rho)$ is the unconditional mean of ndr_t .

Equation (12) has a simple interpretation as the weighted average of the current and the mean levels of ndr_t . As λ approaches unity, which implies a very long time horizon with all future net discount rates weighted equally, the optimal predictor is simply the mean of ndr , μ . In contrast, when the appropriate time horizon is shorter, λ is small and the current value of ndr_t receives more weight. However, because λ is strictly less than one, the optimal weights on the current value of ndr_t and the mean, μ , also depend on the magnitude of the autoregressive parameter, ρ . As ρ approaches unity, the case of a pure random walk, the optimal predictor contains only the current value of ndr_t . In contrast, as ρ approaches zero, the case of a pure white noise process, the optimal predictor contains only the mean, μ .

Table 1
Optimal Forecast Weight on Current ndr for the AR(1) Model

Forecast Horizon	First-Order Autocorrelation Coefficient: ρ					
λ	0.50	0.60	0.70	0.80	0.90	0.95
0.50	0.333	0.429	0.538	0.667	0.818	0.905
0.55	0.310	0.403	0.512	0.643	0.802	0.895
0.60	0.286	0.375	0.483	0.615	0.783	0.884
0.65	0.259	0.344	0.450	0.583	0.759	0.869
0.70	0.231	0.310	0.412	0.545	0.730	0.851
0.75	0.200	0.273	0.368	0.500	0.692	0.826
0.80	0.167	0.231	0.318	0.444	0.643	0.792
0.85	0.130	0.184	0.259	0.375	0.574	0.740
0.90	0.091	0.130	0.189	0.286	0.474	0.655
0.95	0.048	0.070	0.104	0.167	0.310	0.487
0.98	0.024	0.036	0.055	0.091	0.184	0.322

To appreciate the relative weights, Table 1 gives the optimal weights on the current value of ndr_t , the first term in equation (12), for given values of the autocorrelation coefficient, ρ , and the parameter, λ . The parameter λ is labeled "forecast horizon" because, although λ depends in part on the underlying decay parameter, γ and the autocorrelation parameter, ρ , it depends mainly on the forecast horizon, n .⁶ When $\lambda = .70$ and $\rho = .70$, the weight on the current rate is 0.412 and the weight on the mean is .588. Note that for any time horizon, λ , as the degree of autocorrelation increases (higher values of ρ), the weight on the current value increases. For a given degree of autocorrelation, ρ , as the time horizon increases, λ increases, the weight on the current value decreases.

To quantify the gains from using the optimal estimator rather than the long-term average or random walk estimators, we calculate the asymptotic relative efficiency of the optimal estimator using a mean squared error criterion. A large ratio would indicate that the optimal estimator has a much smaller mean squared forecast error compared to the alternative and hence is relatively more efficient. The efficiency of the optimal estimator relative to the long-term average estimator can be shown to be:

$$(13) \quad \frac{E(PNDR_t - \mu)^2}{E(PNDR_t - \overline{PNDR}_t)^2} = \frac{1 - (\rho\lambda)^2}{1 - \rho^2}.$$

⁶The exact correspondence between λ and the forecast horizon, n is given in Tables A-1, A-2 and A-3 in Appendix 1.

If ρ is small and λ is large, that is, if the series is only mildly autocorrelated and the forecast horizon is large, the relative efficiency will be only slightly greater than one, so the efficiency gain from using the optimal estimator would be slight. In contrast, for moderate values of λ and large values of ρ , the relative efficiency can be quite large. In other words, if net discount rates are highly autocorrelated and the forecast horizon is moderate, using the optimal estimator would yield considerable gains. Table 2 presents the relative efficiency gain by using the optimal estimator for a variety of values for λ and ρ .

What do these results imply about the relative asymptotic efficiency of the optimal estimator? Consider a benchmark case: $\rho = .70$, $\gamma = .98$ and a forecast horizon of five years. From Appendix 1, Table A-1 (or directly from equation (9)) these benchmark figures suggest a value for λ of about .70. From Table 1, (or directly from equation (12)) we obtain weights of the current and long-term average of .412 and .588, respectively. That is, the optimal estimator gives approximately equal weight to the random walk and long-term average estimator. From Table 2 (or directly from equation (13)) we see that the MSE of the long-term average is 1.49 times that of the optimal estimator.

The asymptotic efficiency of the optimal estimator relative to the random walk estimator can be expressed as:

$$(14) \quad \frac{E(PNDR_t - ndr_t)^2}{E(PNDR_t - \overline{PNDR}_t)^2} = \frac{2(1 - \rho\lambda)}{(1 + \rho)(1 - \lambda)}.$$

Table 2
Efficiency of the Optimal Estimator Relative to the Long-Term
Average Estimator in the AR(1) Model

Forecast Horizon	First order Autocorrelation Coefficient: ρ					
λ	0.50	0.60	0.70	0.80	0.90	0.95
0.50	1.250	1.422	1.721	2.333	4.197	7.942
0.55	1.233	1.392	1.670	2.240	3.974	7.456
0.60	1.213	1.360	1.615	2.138	3.728	6.924
0.65	1.193	1.325	1.555	2.027	3.462	6.346
0.70	1.170	1.287	1.490	1.907	3.174	5.721
0.75	1.146	1.246	1.420	1.778	2.865	5.050
0.80	1.120	1.203	1.346	1.640	2.535	4.332
0.85	1.093	1.156	1.267	1.493	2.183	3.569
0.90	1.063	1.107	1.183	1.338	1.810	2.759
0.95	1.033	1.055	1.094	1.173	1.416	1.903
0.98	1.016	1.028	1.047	1.088	1.210	1.457

Table 3
Efficiency of the Optimal Estimator Relative to the Random Walk
Estimator in the AR(1) Model

Forecast Horizon	First order Autocorrelation Coefficient: ρ					
λ	0.50	0.60	0.70	0.80	0.90	0.95
0.50	2.000	1.750	1.529	1.333	1.158	1.077
0.55	2.148	1.861	1.608	1.383	1.181	1.088
0.60	2.333	2.000	1.706	1.444	1.211	1.103
0.65	2.571	2.179	1.832	1.524	1.248	1.121
0.70	2.889	2.417	2.000	1.630	1.298	1.145
0.75	3.333	2.750	2.235	1.778	1.368	1.179
0.80	4.000	3.250	2.588	2.000	1.474	1.231
0.85	5.111	4.083	3.176	2.370	1.649	1.316
0.90	7.333	5.750	4.353	3.111	2.000	1.487
0.95	14.000	10.750	7.882	5.333	3.053	2.000
0.98	27.333	20.750	14.941	9.778	5.158	3.026

For shorter time horizons and for highly autocorrelated net discount rate processes, the optimal estimator is marginally more efficient. For longer horizons, and for weakly autocorrelated series, the gain in efficiency can be large. Table 3 shows the relative efficiency gain by using the optimal estimator for a variety of values for λ and ρ . In our benchmark case, we see that the mean forecast error variance of the random walk estimator is twice that of the optimal estimator. Tables 2 and 3 suggest that there are potentially large gains in efficiency from using the optimal estimator.

V. Empirical Example

In this section we present evidence that our proposed estimator would have, on average, worked well for the last 40 years of experience in the U.S. and it would have, on average, worked well for a wide sample of countries over the 1990s. The theoretical results of the previous section suggest that the optimal estimator will outperform the alternatives, but those results rely on the assumptions that the net discount rate process follows a stationary first order autoregressive process with known parameters. In practice, the parameters must be estimated and the process may not be stationary. It is therefore of interest to see how our estimator would have worked when applied to actual net discount rate processes.

We use as our objective function the five-period-ahead average net discount rate,⁷

$$(15) \quad FNDR_{t,5} = \frac{1-\gamma}{1-\gamma^5} \sum_{j=0}^4 \gamma^j (ndr_{t+1+j}).$$

We choose to work with annual data on net discount rates. The net discount rate is defined as $ndr_t = i_t - g_t$, where i_t is the one-year U.S. treasury bill rate and g_t is the annual growth rate of average hourly earnings in manufacturing.⁸ Data on these variables are available from 1954 to 2005. We recursively compute a series of forecasts of the objective function, (15), for each of the years from 1960 to 2000.

Two versions of the long-term average estimator are considered. The first uses the average over the entire sample, from 1954 to the forecast date. The second uses only the average over the previous 10 years prior to the forecast date. The latter reflects the suggestion by Gamber and Sorensen (1994) as well as others that, due to breaks in the data, the long-term average should reflect only more recent observations. The random walk estimator is simply the actual net discount rate at the forecast date.

To compute the optimal estimator, we first examined the time series properties of the net discount rate. Standard criteria (Akaike, Hannan-Quinn, Schwartz) suggest that the data follow a first order autoregressive process. We thus compute the optimal estimator from equation (12). We obtain the value of the weighting parameter, λ , from equation (10) using $\gamma = .98$ (reflecting the long-term average net discount rate of approximately .02.) and $n = 5$ (reflecting the five-year horizon.)⁹

We also evaluate the performance of a simple compromise estimator that is an equally weighted average of the current and long-term net discount rates. This compromise estimator should be asymptotically less efficient, but it has the advantage of simplicity and reflects the essential concept of the optimal estimator: a blending of the two extreme estimators. Further, the theoretical considerations developed in the previous section suggest that an equally weighted average may be close to optimal. (See Table 1)

Our five estimators can each be viewed as weighted averages of the historical data. The long-term average places equal weight on all historical values whereas the 10-year average places equal weight on only the last 10 years. The random walk estimator places all weight on the current value. The compromise estimator, being a simple average of the random walk and long-term average places half of its weight on the current value and the rest on all of the observa-

⁷The choice of a five-year time horizon represents a compromise between the desire to use the most recent experience and the desire to look at long period prediction.

⁸Defining the net discount rate as $(1+i)/(1+g)-1$ yields very similar conclusions. The one-year treasury bill rate is taken to be the rate reported in January of each year. The growth rate of earning is measured as a December to December growth rate.

⁹Note that when calculating the optimal estimator we use the entire sample up to the forecast data. As such the optimal estimator is a combination of the current value and long-term average rather than the 10-year average.

tions. The optimal estimator is also a weighted average of the random walk and long-term average estimators, but the weights are determined by the time-series properties of the net discount rate. Figures 1 to 5 give the predictions of each estimation strategy along with the realization of the future net discount rate, FNDR.

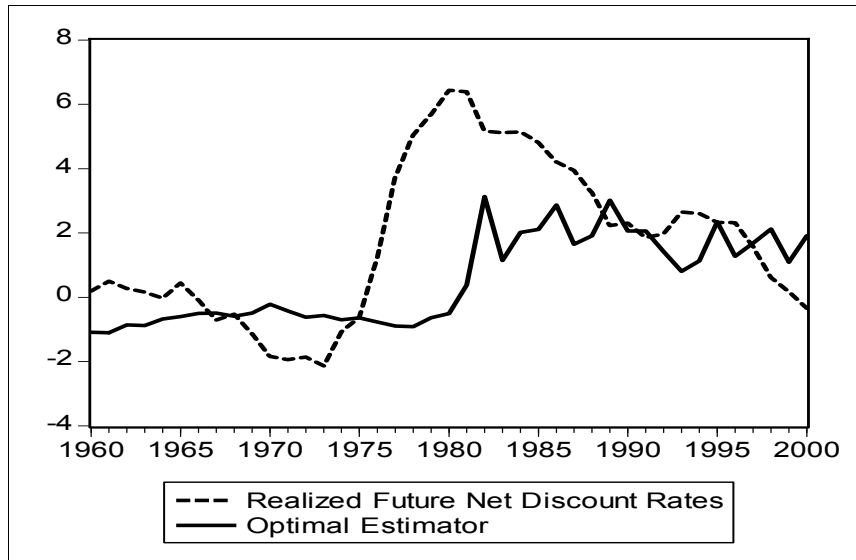


Figure 1. Optimal Estimator vs. Realized Future Net Discount Rates

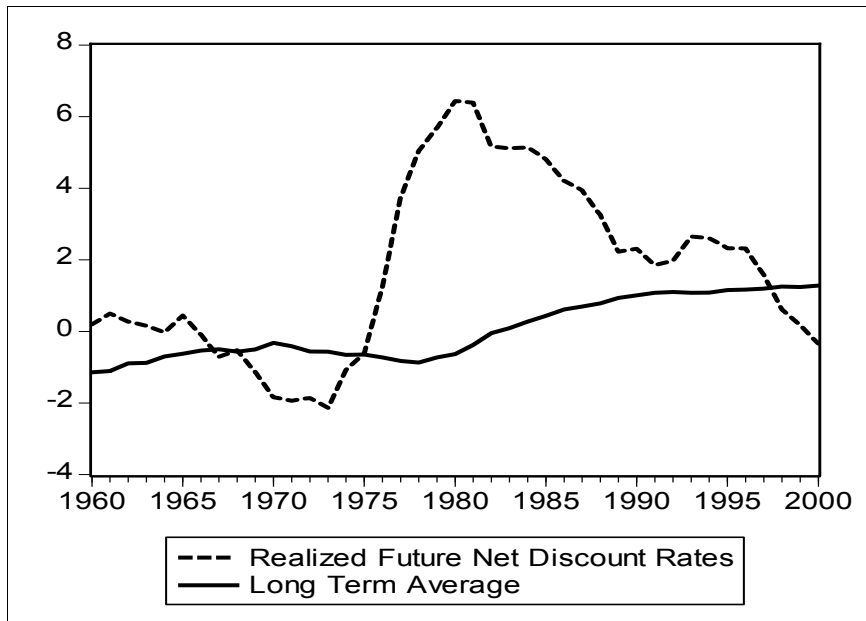


Figure 2. Long-Term Average vs. Realized Future Net Discount Rates

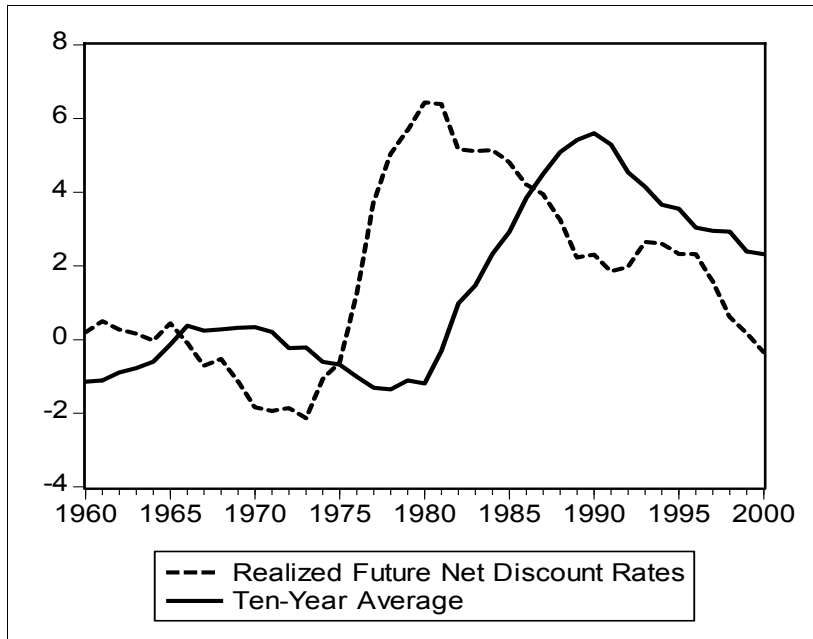


Figure 3. 10-Year Average vs. Realized Future Net Discount Rates

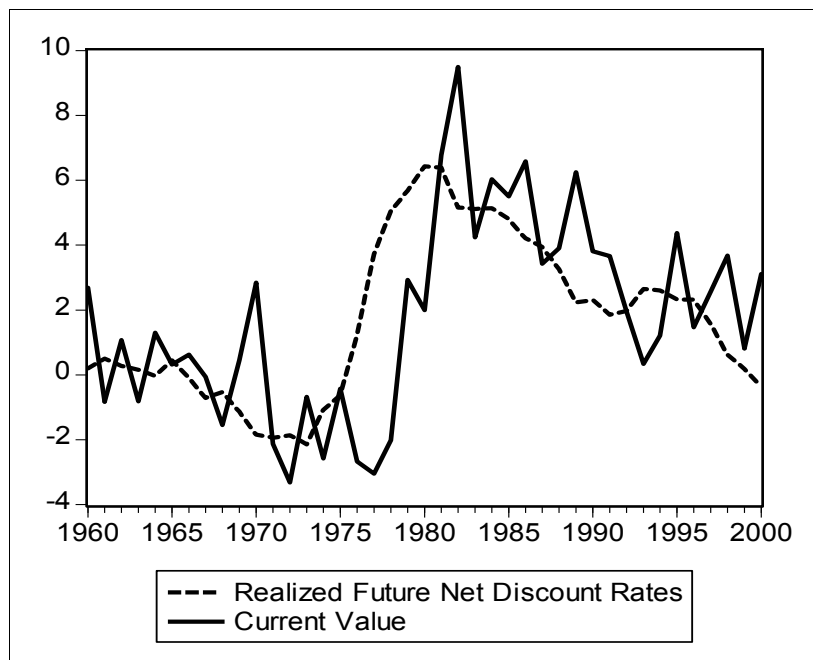


Figure 4. Current Interest Rate vs. Realized Future Net Discount Rates

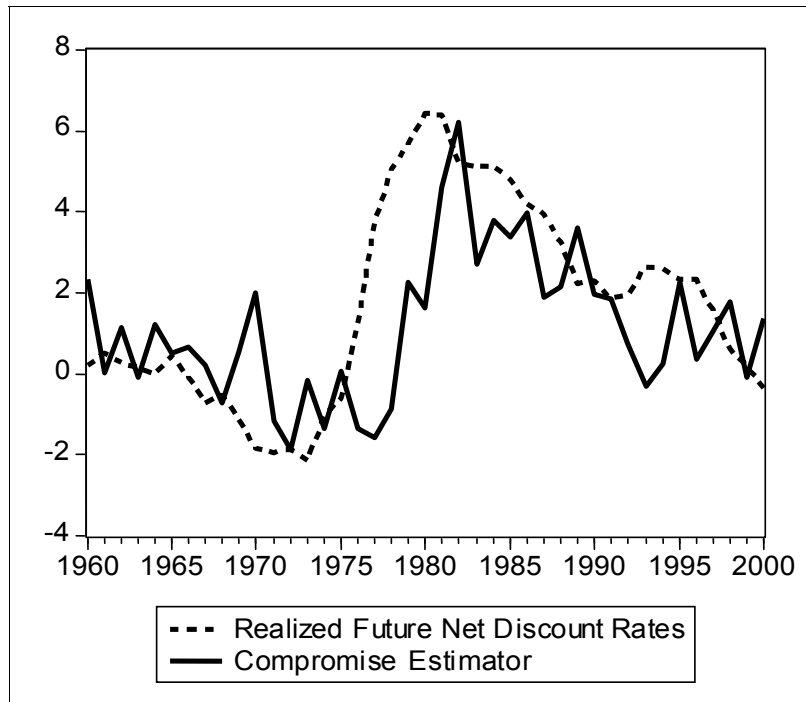


Figure 5. Compromise (Weighted Average) Estimator vs. Realized Future Net Discount Rates

The compromise estimator appears (to us) to track best, but, because it is difficult to discern this from these graphs, we quantify the comparisons according to their mean squared prediction errors (MSPEs). The MSPE is defined as

$$(16) \quad MSPE = \frac{1}{m} \sum_{t=1}^m (\widehat{PNDR}_t - FNDR_{t,n})^2,$$

where m is the sample period of interest, \widehat{PNDR} is the particular net discount rate estimator and $FNDR_{t,n}$ the weighted average of future net discount rates over the subsequent n periods (the realized future net discount rate.)

Mean square prediction errors are shown in Table 4. The optimal estimator performed better than all three of the traditional estimators in the 1960s and overall. The long-term average outperformed the optimal estimator in the 1970s and 1990s, but was worse overall. The 10-year average performed worse than the optimal estimator in all sub-periods and was also worse overall. The random walk estimator beat the optimal estimator in the 1980s, but was slightly worse overall.

Table 4
Mean Squared Prediction Error Post-War U.S. Data

	Estimator				
	Current Value	Long-Term Average	10-Year Average	Optimal Estimator	Compromise Estimator
1960-1969	1.580	0.899	1.111	0.869	1.134
1970-1979	14.706	10.964	13.338	10.989	10.138
1980-1989	6.311	22.197	15.907	13.056	4.466
1990-2000	3.675	1.347	4.895	1.386	2.224
1960-2000	6.498	8.669	8.717	6.449	4.435

The compromise estimator performed surprisingly well, beating our optimal estimator in the 1970s, 1980s and overall. The compromise estimator performed better than the random walk estimator in all four periods and overall. The compromise estimator performed better than the long-term average in the 1970s and 1980s and performed better over the entire time span. It performed better than the 10-year average in all periods except the 1960s.

Although these results are suggestive, the conclusions from Table 4 should not be overstated. The U.S. data contains only eight non-overlapping five-year intervals and even these non-overlapping intervals cannot be considered independent observations. The relative superiority of any of these estimators over a single, short time span can easily be attributed to pure chance. Examining the performance over longer time spans would be desirable, but U.S. data for early periods may be of limited relevance for evaluating current prediction performance.¹⁰

To extend our sample, we examine how our estimators would have worked in a sample 13 countries over the 1990s. The 13 countries are chosen based on the availability of short-term interest rate and wage-rate data in the International Financial Statistics database. We apply the same estimation strategy to each of these countries, modeling the net discount rate process as a first-order autoregressive process.¹¹

Table 5 shows the mean squared prediction error for each estimator across the 13 countries. The current value performed best in two cases, the long-term average in one, the 10-year average in one, the optimal estimator in two and the compromise estimator worked best in the remaining seven. In pair-wise comparisons, the optimal estimator outperformed the random walk estimator in all countries but Greece, Italy and the U.K. The optimal estimator has a lower MSPE than the long-term average in all countries but Korea, and a lower MSPE than the 10-year average in all countries but Germany, Italy, Sweden and the U.K. Overall, the optimal estimator had an average MSE significantly lower than any of the traditional estimators.

¹⁰The imposition and relaxation of wage and price controls during the 1940's imparts wide year-to-year swings in hourly earnings data.

¹¹For 11 of these countries, quarterly data were available. For these countries, we used the interest rate reported in the first quarter and measured the annual growth rate of wages as the first-quarter to first-quarter growth rate. For the remaining two countries, Greece and Norway, we simply take the annual interest rate given and for wage growth use the year-over-year average. The data are described more fully in Appendix 1, Table A-4.

Table 5
Mean Squared Prediction Error 1990-2000 International Data

	Estimator				
	Current Value	Long-Term Average	Ten-Year Average	Optimal Estimator	Compromise Estimator
Australia	8.60	8.11	13.92	1.79	1.38
Canada	3.16	5.36	5.04	1.29	1.15
Finland	18.70	12.87	20.46	9.34	8.08
France	7.29	21.62	15.69	5.22	4.82
Germany	8.20	9.05	0.53	5.52	4.69
Greece	16.14	77.01	38.52	26.82	23.51
Italy	7.16	37.24	9.82	24.45	10.02
Japan	4.05	1.24	3.02	1.18	1.75
Korea	78.52	19.41	35.82	22.05	34.01
Netherlands	4.31	12.23	7.72	2.46	2.85
Norway	10.23	12.32	8.83	6.26	4.13
Sweden	9.11	7.53	5.47	6.21	4.36
U. K.	3.67	8.55	3.10	4.10	1.67
Average	13.78	17.89	12.92	8.98	7.88

Table 6
Relative Mean Squared Prediction Error 1990-2000 International Data

	Estimator				
	Current Value	Long-Term Average	Ten-Year Average	Optimal Estimator	Compromise Estimator
Australia	4.80	4.53	7.78	1.00	0.77
Canada	2.45	4.16	3.90	1.00	0.89
Finland	2.00	1.38	2.19	1.00	0.87
France	1.40	4.14	3.01	1.00	0.92
Germany	1.49	1.64	0.10	1.00	0.85
Greece	0.60	2.87	1.44	1.00	0.88
Italy	0.29	1.52	0.40	1.00	0.41
Japan	3.42	1.05	2.55	1.00	1.48
Korea	3.56	0.88	1.62	1.00	1.54
Netherlands	1.75	4.97	3.14	1.00	1.16
Norway	1.63	1.97	1.41	1.00	0.66
Sweden	1.47	1.21	0.88	1.00	0.70
U. K.	0.90	2.09	0.76	1.00	0.41
Average	1.98	2.49	2.24	1.00	0.89

The compromise estimator performed surprisingly well. It had a lower MSPE than the random walk estimator in all countries but Greece and Italy and out-performed the long-term average in every country but Japan and Korea. It performed better than the 10-year average in all countries but Germany and Italy. The MSPE for the compromise estimator was lower than the optimal estimator in all but three of the countries and had a slightly lower average MSPE across all countries.

Differences in the volatility of net discount rates across countries complicate the comparisons of relative efficiency. To ease the comparison, Table 6 presents each estimator's MSPE relative to the mean squared error for the optimal estimator. This corrects for different scales across countries. Overall, the long-term average and 10-year average estimators had MSPEs at least twice as large as that observed for the optimal estimator. The random walk estimator had a smaller divergence from the optimal estimator; but even here the relative MSPE was twice that of the optimal estimator. Again, the simple compromise estimator was seen to perform surprisingly well, outperforming the optimal estimator in average relative mean squared prediction error.

V. Conclusion

There is an ongoing debate in forensic economics about the time series nature of net discount rates. The debate has implications for the appropriateness of estimators to use in predicting future net discount rates. If the net discount rate series is stationary, estimates based on historic values are reasonable. Alternatively, if the series exhibits a unit root, then the best predictor of the next period's discount rate depends mainly on the current net discount rate.

We derive a general, optimal estimator of future net discount rates that depends on the forecast horizon and the time-series properties of the net discount rate process. In the important and empirically relevant special case where the net discount rate follows a first-order autoregressive process, the optimal estimator is a simple weighted average of two unit root and long-term average estimators. The weights depend on both the length of the forecast horizon and the degree of autocorrelation in the net discount rate process. For short forecast horizons and when net discount rates are close to a random walk, the optimal estimator is weighted more heavily toward the current value. For long forecast horizons and when the degree of autocorrelation is low, the estimator is weighted more heavily toward the long-term average.

We then compare the asymptotic performance of the optimal estimator to the random walk and long-term average estimators. Our benchmark estimates suggest that the mean squared forecast error using either the long-term average or the current rate can be almost twice that of the optimal estimator.

Historical U.S. and international net discount rates are then used to compare the actual performance of the optimal and extreme estimators. For the U.S. data, the optimal estimator generally outperformed the traditional estimators. These results held true across a sample of 13 developed countries. On average the optimal estimator appears approximately twice as efficient as either the current value or long-term average.

Because calculating and justifying the weights in the optimal estimator can be problematic, we also suggest a compromise estimator that equally weights the current and long-term average net discount rates. Empirical results show that this compromise estimator outperforms either the long-term average or the current value and performs very much like the optimal estimator. This may be an appealing solution that has both theoretical support and

can be easily implemented and justified in a forensic setting. Sometimes the "truth" actually does lie somewhere between the two extremes.

References

- Braun, B., J. Lee, and M.C. Strazicich, *Historical Net Discount Rates and Future Economic Losses: Refuting the Common Practice, Economic Foundations of Injury and Death Damages*, edited by Kaufman, R. T., J. D. Rodgers, and G. D. Martin, Edward Elgar Publishing Ltd., 2005, 468-491.
- Ewing, B. T., J. T. Payne, and M. J. Piette, "Time Series Behavior of Medical Cost Net Discount Rates: Implications for Total Offsetting and Forecasting," *Journal of Forensic Economics*, Winter 2001, 14(1), 53-61.
- _____, _____, and _____, "Forecasting Medical Net Discount Rates," *Journal of Risk and Insurance*, 2003, 70(1), 85-95.
- Gamber, E. N., and R. L. Sorensen, "On Testing for the Stability of the Net Discount Rate," *Journal of Forensic Economics*, 1993, 7, 9-79.
- _____, and _____, "Are Net Discount Rates Stationary? The Implications for Present Value Calculations: Comment," *Journal of Risk and Insurance*, 1994, 61, 503-512.
- Haslag, J. H., M. Nieswiadomy, and D.J. Slottje, "Are Net Discount Rates Stationary? The Implications for Present Value Calculations," *Journal of Risk and Insurance*, 1991, 58, 505-512.
- _____, _____, and _____, "Are Net Discount Rates Stationary? Some Further Evidence," *Journal of Risk and Insurance*, 1994, 61(3), 513-518.
- Hays, P., M. Schreiber, J.E. Payne, B. T. Ewing and M. J. Piette, "Are Net Discount Ratios Stationary? Evidence of Mean Reversion and Persistence," *Journal of Risk and Insurance*, 2000, 67(3), 439-449.
- Horvath, P. A., and E. L. Sattler, "Calculating Net Discount Rates—It's Time to Recognize Structural Changes: A Comment and Extension," *Journal of Forensic Economics*, 1997, 10, 327-332.
- Johnson, W. D., and G. M. Gelles, "Calculating Net Discount Rates Rates—It's Time to Recognize Structural Change," *Journal of Forensic Economics*, 1996, 9, 119-129.
- Lee, J., and M. Strazicich, "Minimum LM Unit Root Test with Two Structural Breaks," *The Review of Economics and Statistics*, 2003, 85(4), 1082-1089.
- Payne, J. E., B. T. Ewing and M. J. Piette, "An Inquiry Into the Time Series Properties of Net Discount Rates," *Journal of Forensic Economics*, 1999, 12(3), 215-223.
- _____, _____, and _____, "Mean Reversion in Net Discount Rates," *Journal of Legal Economics*, Spring/Summer 1999, 9(1), 69-80.
- Perron, P., "Further Evidence on Breaking Trend Functions in Macroeconomic Variables," *Journal of Econometrics*, October 1997, 80(2), 355-85.
- Sargent, T. J., *Macroeconomic Theory*, Orlando, FL: Academic Press Inc., 1987.
- Sen, A., G. M. Gelles and W. D. Johnson, "A Further Examination Regarding the Stability of the Net Discount Rate," *Journal of Forensic Economics*, Winter 2000, 13(1), 23-28.
- _____, _____, and _____, "Structural Instability in the Net Discount Rate Series Based on High Grade Municipal Bond Yields," *Journal of Legal Economics*, Fall 2002, 12(2), 87-100.
- Shiller, R. J., "The Volatility of Long-Term Interest Rates and Expectations Models of the Term Structure," *Journal of Political Economy*, Dec.1979, 87(6), 1190-1219.

Table A-3
Value of λ for $\gamma = .96$

Periods (n)	First order autocorrelation coefficient: ρ					
	.50	.60	.70	.80	.90	.95
2	0.39	0.38	0.36	0.35	0.34	0.33
3	0.58	0.56	0.54	0.53	0.51	0.50
4	0.68	0.67	0.65	0.63	0.61	0.60
5	0.75	0.73	0.72	0.70	0.68	0.67
10	0.87	0.87	0.86	0.85	0.83	0.82
15	0.91	0.91	0.90	0.90	0.88	0.87
20	0.93	0.92	0.92	0.92	0.91	0.90
30	0.94	0.94	0.94	0.94	0.93	0.93
50	0.95	0.95	0.95	0.95	0.95	0.95

Table A-4 presents the data used in the international comparisons used in the empirical section.

Table A-4
Description of Data used for International Comparisons

Country	Short Term Interest Rate Measure	Wage or Earnings Measure	Data Range
Australia	Money Market Rate	Average Weekly Earnings	Quarterly 1967:1-2005:4
Canada	Treasury Bill Rate	Average Hourly Earnings (Man.)	Quarterly 1960:1-2005:4
Finland	Money Market Rate	Hourly Earning Index	Quarterly 1960:1-2005:4
France	Call Money Rate	Labor Costs Index	Quarterly 1960:1-2005:4
Germany	Call Money Rate	Hourly Earnings Index	Quarterly 1962:1-2005:4
Greece	Bank Deposit Rates	Average Monthly Earnings	Annual 1962-2005
Italy	Treasury Bill Rate	Contractual Wage Rates	Quarterly 1960:1-2005:4
Japan	Call Money Rate	Average Monthly Earnings	Quarterly 1960:1-2005:4
Korea	Money Market Rate	Average Monthly Earnings	Quarterly 1960:1-2005:4
Netherlands	Call Money Rate	Average Hourly Wage Rates	Quarterly 1960:1-2005:4
Norway	Central Bank Discount Rate	Average Monthly Earnings	Annual 1962-2005
Sweden	Bank Deposit Rate	Average Hourly Earning (Man.)	Quarterly 1961:1-2005:4
U. K.	Treasury Bill Rate	Average Monthly Earnings	Quarterly 1963:1-2005:4

Appendix 2

We seek an expression for the predictor of the true truncated objective function,

$$(A2-1) \quad \widehat{FNDR}_{t,n} = \frac{1-\gamma}{1-\gamma^n} \sum_{j=0}^{n-1} \gamma^j (\widehat{ndr}_{t+1+j}),$$

where the hat denotes the (least squares) prediction based on information on net discount rates at time t and before.

We assume net discount rates follow a first-order process,

$$(A2-2) \quad ndr_t = a + \rho ndr_{t-1} + e_t,$$

with known coefficients. The optimal predictor of net discount rates at time $t+1+j$ is given by,

$$(A2-3) \quad \widehat{ndr}_{t+1+j} = \left[a \frac{1-\rho^{j+1}}{1-\rho} + \rho^{j+1} ndr_t \right].$$

Substituting (A2-3) into (A2-1) gives,

$$(A2-4) \quad \widehat{FNDR}_{t,n} = \frac{1-\gamma}{1-\gamma^n} \sum_{j=0}^{n-1} \gamma^j \left[a \frac{1-\rho^{j+1}}{1-\rho} + \rho^{j+1} ndr_t \right].$$

Solving the geometric sums in (A2-4) gives,

$$(A2-5) \quad \widehat{FNDR}_{t,n} = \frac{1-\gamma}{1-\gamma^n} \left[\frac{\rho(1-(\gamma\rho)^n)}{1-\gamma\rho} ndr_t + \frac{a}{1-\rho} \frac{(1-\gamma^n)}{1-\gamma} - \frac{\rho a}{1-\rho} \frac{1-(\gamma\rho)^n}{1-\gamma\rho} \right].$$

Rearranging and simplifying (A2-5) gives,

$$(A2-6) \quad \widehat{FNDR}_{t,n} = \frac{\rho(1-\gamma)(1-(\gamma\rho)^n)}{(1-\gamma\rho)(1-\gamma^n)} ndr_t + \left(\frac{1}{1-\rho} - \frac{\rho(1-\gamma)(1-(\gamma\rho)^n)}{(1-\rho)(1-\gamma\rho)(1-\gamma^n)} \right) a,$$

which is equation (7) in the text.

Now, letting n go to infinity, (assuming $|\gamma| < 1$ and $|\rho| < 1$), we have

$$(A2-7) \quad \widehat{FNDR}_{t,n} = \frac{\rho(1-\gamma)}{(1-\gamma\rho)} ndr_t + \left(\frac{1}{1-\rho} - \frac{\rho(1-\gamma)}{(1-\rho)(1-\gamma\rho)} \right) a.$$

Simplifying (A2-7) gives,

$$(A2-8) \quad \widehat{PNDR}_t = \frac{\rho(1-\lambda)}{1-\rho\lambda} ndr_t + \frac{1}{1-\rho\lambda} a,$$

which is equation (8) in the text.